Chapter One

Patterns

10 HOURS

Introduction

To find and extend patterns is likely the most important problem-solving strategy. In fact, mathematics itself has been called the study of patterns. In this chapter, students find, describe, and extend patterns in numbers and spatial patterns. They are introduced to sequences and how to write them, and to the properties of special sequences (including Fibonacci and arithmetic sequences), power sequences (quadratic and cubic sequences), and geometric sequences. To identify these types of sequences and distinguish between them, students will use two main strategies:

- *Common differences between successive terms in a sequence:* For example, if t_n stands for the terms of a sequence, the students calculate and may find that $t_5 - t_4 = t_4 - t_3 = t_3 - t_2$, and so on.
- *Graphing the sequences: S*tudents will be able to look at the graphs and see how the shapes of the graphs of the sequences are the same and how they differ.

Students should be encouraged to use graphing technology to find rules for arithmetic, quadratic, and other sequences. However, they need to look at data, find patterns, and describe relationships to be able to make generalizations about extending terms in a sequence. It is most important for them to use words to describe rules and relationships, although in many cases, they will be able to use symbols for these same rules and relationships.

Students will learn to use the constant term in a sequence of first-level differences, D_1, to make a rule for building an arithmetic sequence, and to use the extension of patterns to make a rule for building simple quadratic sequences. They will look closely at sequences that come from quadratic relations, as these are the simplest examples of power sequences. The next chapter, Quadratics, builds on what is learned in this section. Geometric sequences are presented informally to contrast their properties with arithmetic, quadratic, and cubic sequences.

Connections

This chapter builds on students' knowledge of linear and non-linear relations, in particular, quadratic relations. Students will use sequences of first- and second-level differences and graphing to distinguish between linear and quadratic relations, as they will see in Chapter 2, Quadratics.

1.1 *Patterns in Sequences*

Suggested instruction time: 3 hours

Purpose of the Section

Students will look at properties of number sequences, and solve sequence problems by identifying and extending patterns. They will develop the Fibonacci sequence and look at a few of its many properties. They will also be introduced to a finite and an infinite sequence and the notation that goes with them. They will graph the term number and term value of some sequences, and use the graph to figure out the slope, if the graph is linear.

CURRICULUM OUTCOMES (SCOs)	RELATED ACTIVITIES	STUDENT BOOK
■ determine and describe patterns and use them to solve problems C5	■ find and extend patterns	p. 2
	■ solve problems by identifying and extending patterns in spatial and number sequences	p. 6
■ explore, describe, and apply the Fibonacci sequence C6	■ find and explore the properties of a Fibonacci sequence	p. 8

ASSUMED PRIOR KNOWLEDGE

- using technology to graph linear and non-linear relations
- distinguishing between linear and non-linear relationships
- calculating slope and *y*-intercept of a line and using this information to determine the linear relation

NEW TERMS AND CONCEPTS	PAGE
■ terms of a sequence	2
■ finite and infinite sequences	5
■ Fibonacci sequence	9

Suggested Introduction

Before starting Investigation 1, ask students to read page 2 of the Student Book and discuss why being able to identify and extend patterns is important. They might also identify patterns in the classroom; for example, tiling patterns on the floor or ceiling, or geometrical patterns on clothing. If a student knows another language, have him or her write and say the number words from "one" to "twenty." Challenge the class to look for patterns that might help a beginning speaker learn to say and write the number names.

Investigation 1

Finding and Extending Patterns

[Suggested time: 60 min] *[Text page 2]*

Purpose

Students will identify and extend the spatial and number patterns in building shapes and solids. They will solve problems by identifying and extending both geometric and numeric patterns. The following concepts are introduced: sequence, term of a sequence, infinite and finite sequences, and ellipsis. Students will also use the shapes of graphs to distinguish between different types of sequences. They will calculate the slopes of linear graphs and compare their value to the terms in a sequence of differences.

Management Suggestions and Materials

Have students read about the brick sequence in the introduction to Section 1.1. Discuss any patterns in the first three brick towers. Point out that each row must have an odd number of bricks. Have students use a pattern to describe how to build a tower with four rows.

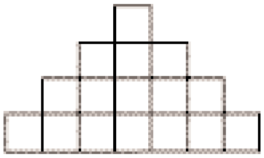
Students will need:

- grid paper or graphing technology
- cube-a-links (optional)

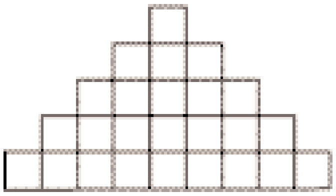
Procedure

Steps A and B

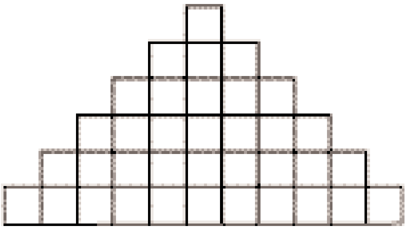
Give students cube-a-links to build the towers, or square paper to draw two-dimensional diagrams.



4 rows and 16 bricks



5 rows and 25 bricks



6 rows and 36 bricks

Suggest that students complete a table to help them spot the pattern.

Rows	1	2	3	4	5	6
Bricks	1	4	9	16	25	36

Notebook Entry

Have students copy the definitions and terminology used with sequences, and write examples to illustrate each. After the term "ellipsis" is introduced in Question 10, they can record the definition of this term as well.

term—each item in a sequence

sequence—an arrangement of numbers, symbols, or pictures in order, where each item or term follows another according to a rule. A sequence is said to be *finite* if it eventually ends. If the sequence continues endlessly, it is said to be *infinite.*

Did You Know?

Students interested in visual and brick designs might want to study the brick wall designs at **www.historiccharlotte.org/kids/Guidebook/brick.html**

Note

Differences between successive terms and how they are used to distinguish between arithmetic and power sequences will be presented formally in Investigation 3.

Think about …

Step C

Students should understand that the 10th term in the sequence in Step C is the number of bricks needed to build a tower with 10 rows: in this case, 100 bricks. To describe how to find the number of bricks in a tower of 50 rows, some students might use a recursive pattern that repeatedly adds differences, while others might use the non-recursive pattern or rule, bricks = rows × rows, and calculate the number of bricks as $50 \times 50 = 2500$.

The sequence in Step B is simply the first six values for the number of bricks: $\{1, 4, 9, 16, 25, 36\}$.

Step C

Students will likely see at least two patterning rules.

- Some students will likely find differences between the terms in the sequence for the number of bricks:

  $\{1, 4, 9, 16, 25, 36, \ldots\}$

  3 5 7 9 11

  These differences can be carried further to show a sequence to 10 terms.

  $\{1, 4, 9, 16, 25, 36, 36 + 13 = 49, 49 + 15 = 64, 64 + 17 = 81, 81 + 19 = 100, \ldots\}$

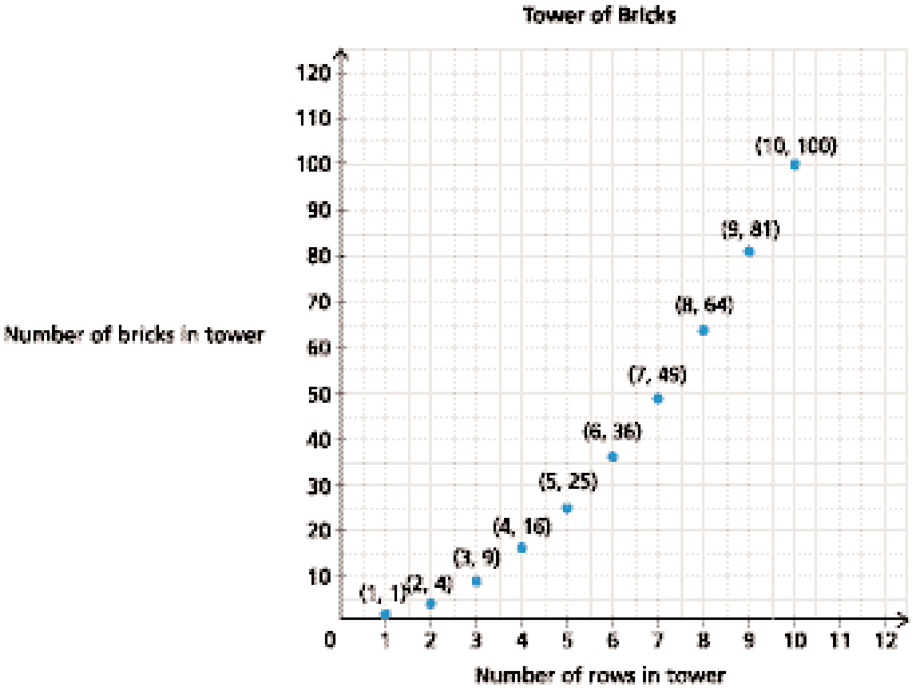
  3 5 7 9 11 13 15 17 19

- Other students will likely recognize that each term in the sequence is a square number and the sequence can be carried on as shown.

  $\{1, 4, 9, 16, 25, 36, \ldots\} \rightarrow \{1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2, 9^2, 10^2, \ldots\}$

Steps D and E

Students can use either grid paper or graphing technology for the graph, which should be in the shape of part of a parabola as shown:



Have students predict where the next point on the graph will be. Some students might remember from *Constructing Mathematics*, *Book 1*, that the graph in Step D is part of the graph for the quadratic relation $y = x^2$.

Step F

Students should see that the graph is not a straight line and, therefore, there is not a linear relationship between the number of rows and number of bricks in a tower. Use the graph to review slope. If the graph were linear, the slope would be the same regardless of the points chosen. However, the number of bricks, or the y-value, doesn't increase by a constant number each time the term number, or the

x-value, increases by 1. This lack of constant increase in y for a constant difference in x means that the relationship is non-linear.

Investigation Questions

QUESTION 1 Page 3

Students who recognize the rule $b = r^2$, where b is the number of bricks and r is the number of rows, will find the answers quickly. Others will use the fact that each time a new row is added, the total number of bricks increases by the next odd number.

Answers

1. (a) 400 bricks (b) 625 bricks (c) 10 000 bricks

QUESTION 2 Page 3

Students can draw the tower from the top down by including one brick on the top row, three bricks on the second row from the top, and so forth until they have created a tower with 144 bricks.

Those who recognize the rule $b = r^2$, where b is the number of bricks and r is the number of rows, can reverse the reasoning and use the rule $r = \sqrt{b}$.

Answer

2. 12 rows

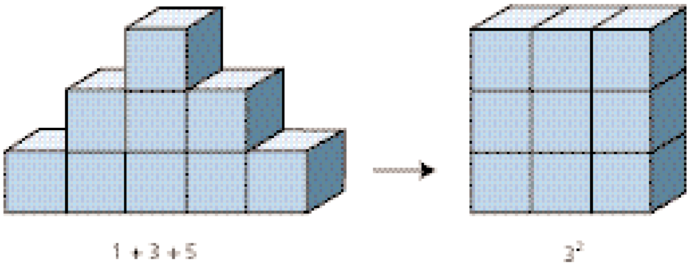
QUESTION 3 Page 3

Students can draw the tower to note that each row contains an odd number of bricks from 1 to 19. Therefore, the total number of bricks in the tower is the sum of the odd numbers from 1 to 19, inclusive.

Each sum of consecutive odd numbers can be expressed as a square number.

$1 = 1^2$
$1 + 3 = 2^2$
$1 + 3 + 5 = 3^2$
$1 + 3 + 5 + 7 = 4^2$
$1 + 3 + 5 + 7 + 9 = 5^2$
$1 + 3 + 5 + 7 + 9 + \ldots + 19 = 10^2$

Students using cube-a-links might show the generalization visually by rearranging a tower to form a square as shown.



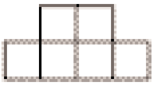
Answer

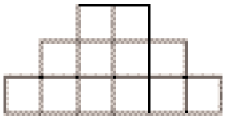
3. equivalent

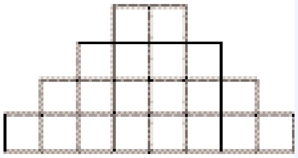
Challenge Yourself

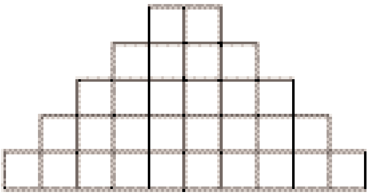
(a) and (b) The first six towers and the number of bricks represented by six terms in a sequence are shown.

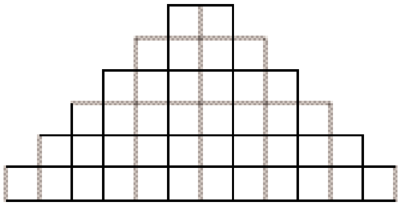

1 row and 2 bricks


2 rows and 6 bricks


3 rows and 12 bricks


4 rows and 20 bricks


5 rows and 30 bricks


6 rows and 42 bricks

Sequence: {2, 6, 12, 20, 30, 42}

(c) Students will either draw the seventh to tenth towers or add consecutive even numbers to the previous terms in the sequence. For example, {2, 6, 12, 20, 30, 42, 42 + 14, 56 + 16, 72 + 18, 90 + 20, …} = {2, 6, 12, 20, 30, 42, 56, 72, 90, 110}.

Some students might use differences to extend the pattern:

{2, 6, 12, 20, 30, 42, 42 + 14,
4 6 8 10 12 14 16
56 + 16, 72 + 18, 90 + 20, …}
18 20
= {2, 6, 12, 20, 30, 42, 56, 72, 90, 110, …}.

(d) The tenth term in the sequence is 110 and, therefore, the number of bricks in a tower with an even number of bricks in each of the 10 rows is 110.

convex polygon—a **polygon** whose interior angles are all less than or equal to 180°

polygon—a closed shape made from connected line segments (sides)

vertex—a corner of a polygon

Management Tip

Throughout this chapter, have the students use words to express rules and relationships. Then help them to use symbols for these same rules and relationships.

Think about ...

Question 4

Students should see, or discover, that it does not matter which vertex the lines start at, since the number of triangles remains the same.

Check Your Understanding

[Completion and discussion: 60 min]

QUESTION 4 Page 3

Have students make a table such as the following:

Sides	3	4	5	6	7	8	9	10
Triangles	1	2	3					

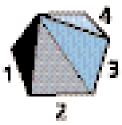
Most students should readily see the pattern that the number of inscribed triangles is two less than the number of sides. Have them say the relationship in words: "The number of triangles can be found by subtracting two from the number of sides."

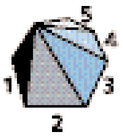
Once students can use words for the relationship, have them use variables to show the relationship. For example, $t = s - 2$, where s is the number of sides of a convex polygon and t is the number of triangles that can be made by drawing lines from only one vertex to every other vertex.

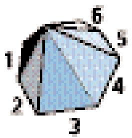
Before students make a scatter plot, have them predict the shape of the graph from the formula $t = s - 2$. In part (e), you might review how to graph a linear relation by finding the slope and y-intercept, which was introduced in *Constructing Mathematics, Book 1.* In this case, the graph has a slope of 1 and a y-intercept of -2.

Answers

4. (a)



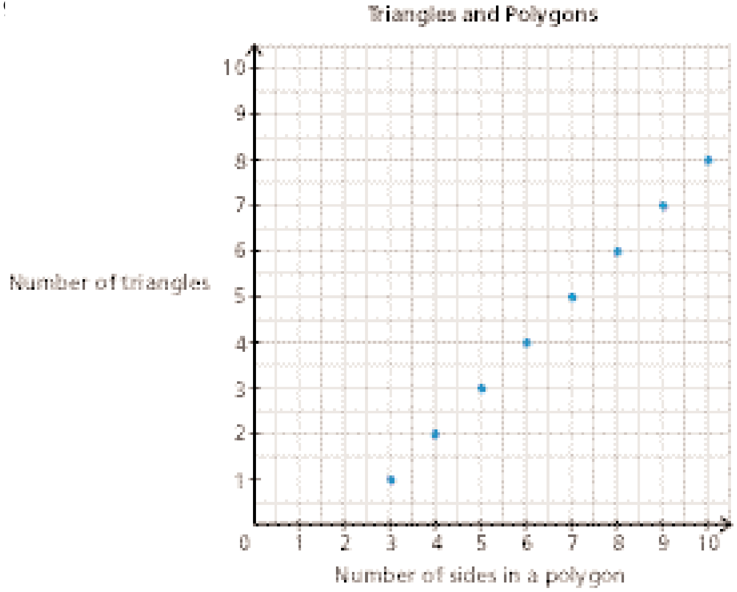




(b) 10 − 2 = 8

(c) Subtract 2 from 50 to get 48 triangles.

(d) The graph is linear.



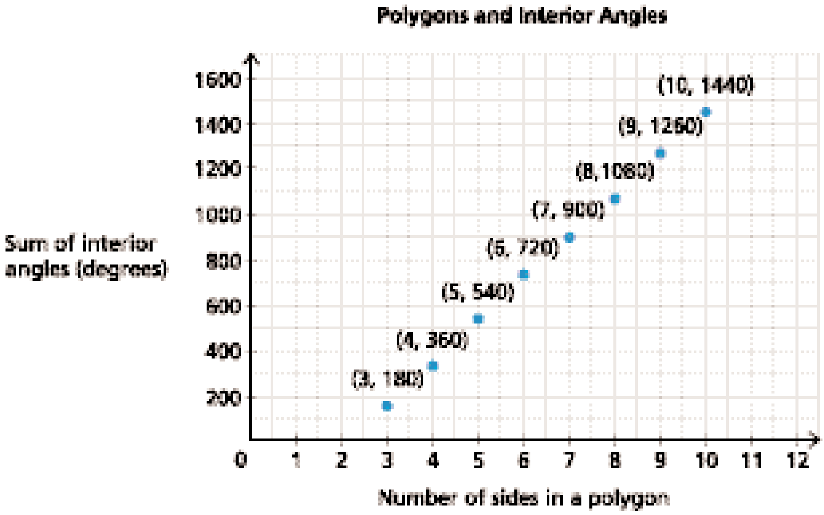
(e) *example:* The line has a slope of 1 and a y-intercept of -2 (found by extending the line to the y-axis). Therefore, the relationship is $t = s - 2$, where s is the number of sides of a polygon and t is the number of triangles.

QUESTION 5 Page 4

Students can use a pattern in the table to find the relationship between the sum of the interior angles and the number of sides in a polygon. Students should see that, as a side is added to a previous polygon, the sum of the interior angles increases by 180°.

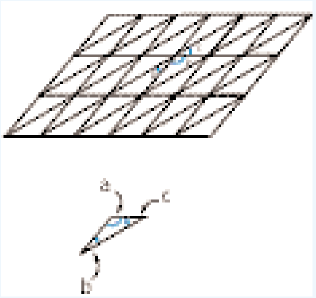
Number of sides	3	4	5	6	7	8	9	10
Number of triangles	1	2	3	4	5	6	7	8
Sum of interior angles (degrees)	180	360	540	720	900	1080	1260	1440

The graph shows the relationship between the sum of the interior angles and the number of sides in a polygon.



Management Tip

For Question 5, remind students that the sum of the interior angles in a triangle is 180°. Students can confirm this by tessellating various types of triangles (drawing identical triangles laid out in a pattern with no gaps or overlapping). For example, the following tessellation was created from congruent scalene triangles. It shows that the interior angles *a*, *b*, and *c* form a straight line at the vertex of the tessellation. In other words, the sum of the interior angles of the triangle is 180°.



slope—the steepness of a line, calculated by comparing the rise and run:

$$\text{slope} = \frac{\text{amount of rise}}{\text{amount of run}}$$

Students can see from the graph that the relationship is linear. Students should see that the slope of the line is 180 because the difference between y-values for consecutive polygons is 180 for each x-value increase of one side of a polygon.

Part (e) informally introduces sequences of differences, which will be treated more formally in the next section.

Sequence: {180, 360, 540, 720, 900, 1080, 1260, 1440}

Sequence of differences: {360 − 180, 540 − 360, 720 − 540, 900 − 720, ...}
= {180, 180, 180, 180, ...}

Students should see that the constant term 180 is identical to the slope of the line in the graph. In the next section, they will be able to look further at the relationship between the slope of a graph and the sequences of differences.

Answers

5. (a) *example:* The sum of the interior angles increases by 180° as a side is added to a polygon; the difference between each pair of adjacent columns in the third row is 180.
   (b) See the previous discussion.
   (c) See the graph and the previous discussion.
   (d) 180
   (e) The differences are each 180, which is equal to the slope of the line. Because you are changing the x-value by one, you are also changing the y-value by a constant amount, 180.

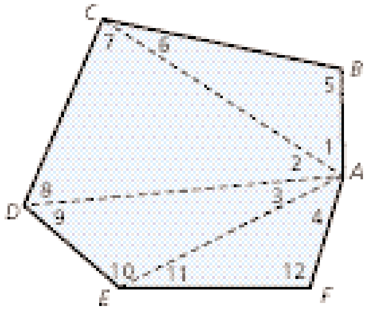
QUESTION 6 Page 4

This question helps students connect the results of Questions 4 and 5, and to see the relationship between the number of sides and the sum of the interior angles in a convex polygon.

In comparing the second and third rows in the table from Question 5, students should realize that each sum can be found by multiplying the number of triangles by 180.

From Question 4, students can see that the number of triangles can be found by subtracting 2 from the number of sides of the polygon. Therefore, the sum of the interior angles of a polygon can be found by subtracting 2 from the number of sides and then multiplying by 180. Students might express this relationship with words or using symbols, such as, "The sum of the interior angles of an n-sided polygon is $(n - 2) \times 180$."

You can help students make sense of this relationship through a discussion about "triangulating" any polygon, such as a hexagon divided into four triangles.



The diagram shows that the sum of the interior angles of the hexagon equals the sum of the interior angles of the four triangles. That is
$\angle A = \angle 1 + \angle 2 + \angle 3 + \angle 4$; $\angle B = \angle 5$;
$\angle C = \angle 6 + \angle 7$; $\angle D = \angle 8 + \angle 9$;
$\angle E = \angle 10 + \angle 11$; and $\angle F = \angle 12$.

Because the sum of the interior angles of a triangle equals 180°, and the hexagon contains the 12 angles formed by 4 triangles, we can see that the sum of the interior angles of a hexagon equals $4 \times 180° = 720°$.

In general, a polygon with n sides can be triangulated (as seen in Question 4) into $n - 2$ triangles and, therefore, the sum of the interior angles of an n-sided polygon is $(n - 2) \times 180°$.

Have students discuss why the relationship can also be $180n - 360$, and have them graph this on a graphing calculator. They can then note that the graph is identical to the one they created in Question 5(c).

Answers

6. (a) and (b) See the previous discussion.
   (c) $(40 - 2)180° = 6840°$

QUESTION 7 Page 4

Students draw their staircases on grid paper. Some students might want to build the staircases using cube-a-links and record the results in the table.

Number of steps	1	2	3	4	5	6	7	8	9	10
Number of blocks	1	3	6	10	15	21				

Most students will probably find a recursive pattern using differences between successive numbers of blocks to extend the table or sequence:

{1, 3, 6, 10, 15, 21, …} →
2 3 4 5 6

{1, 3, 6, 10, 15, 21, 21 + 7 = 28, 28 + 8 = 36, 36 + 9 = 45, 45 + 10 = 55}
2 3 4 5 6 7 8 9 10

Number of steps	1	2	3	4	5	6	7	8	9	10
Number of blocks	1	3	6	10	15	21	21 + 7	28 + 8	36 + 9	45 + 10
Differences		2	3	4	5	6	7	8	9	10

Some students might discover the non-recursive pattern by comparing adjacent numbers of steps and the number of blocks. For example, when 4 and 5 are compared to 10, the pattern is $\frac{1}{2}$ of $4 \times 5 = 10$.

Number of steps	1	2	3	**4**	**5**	6	7	8	9	10
Number of blocks	1	3	6	**10**	15	21				

If desired, this pattern can be developed into a general rule such as the following: *number of blocks* $= \frac{s(s+1)}{2}$, where s is the number of steps. As a result, some students may need some assistance in arriving at the general pattern. However, allow students to try it on their own first.

You could also have students use the blocks to describe the formula. If they take two sets of blocks, each forming the pattern for n steps, these two sets of steps can be fitted together along their diagonal edges to form a rectangle with dimensions

Management Tip

After students finish Question 6, discuss why it is good to create a general rule to express relationships.

Challenge Yourself

Once students realize that the ending digits repeat in groups of four, suggest that they arrange the digits and powers of 2 in columns of four, as shown:

2 4 8 6 2^1 2^2 2^3 2^4
2 4 8 6 2^5 2^6 2^7 2^8
2 4 8 6 2^9 2^{10} 2^{11} 2^{12}
⋮
2 4 8 **6** 2^{37} 2^{38} 2^{39} $\mathbf{2^{40}}$
2 **4** 8 6 2^{41} $\mathbf{2^{42}}$ 2^{43} 2^{44}

This arrangement will help them to realize that any power of 2 that is a multiple of 4 (such as 4, 8, 12, and so forth) will have an ending digit of 6. Therefore, 2^{40} will end in 6, while 2^{42} will end in 4.

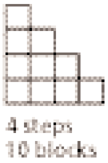
Likewise, students might arrange the ending digits of powers of 3 into columns of four because the ending digits repeat as 3, 9, 7, 1.

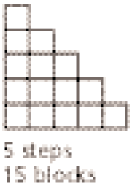
(a) 2, 4, 8, 6, 2, 4, 8, 6
(b) {2, 4, 8, 6, 2, 4, 8, 6, …}
(c) *example.* The ending digits repeat as 2, 4, 8, and 6.
(d) 2^{40} ends in 6, and 2^{42} ends in 4.
(e) 3, 9, 7, 1, 3, 9, 7, 1; {3, 9, 7, 1, 3, 9, 7, 1, …}; *example:* The ending digits repeat as 3, 9, 7, and 1.
(f) 3^{25} ends in 3; 3^{48} ends in 1

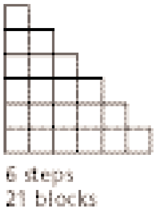
n blocks × $(n + 1)$ blocks. The number of blocks in the rectangle is, therefore, $n(n + 1)$. Thus, the number of blocks in each set of n steps is $\frac{1}{2}n(n + 1)$.

Answers

7. (







(b) See the completed table in the previous discussion.

(c) {1, 3, 6, 10, 15, 21, …} becomes {2, 3, 4, 5, 6, …}; *example:* the counting numbers from 2 onward

(d) 55 blocks; number of blocks $= \frac{10(10 + 1)}{2} = 55$

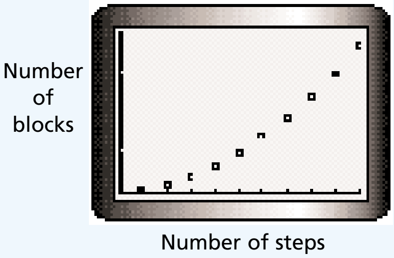
(e) *example:* To find the number of blocks in a staircase with 11 steps, you can add 11 to the 55 blocks in a staircase with 10 steps. Then, you add 12, 13, and so on until you find the number of blocks in a staircase with 50 steps. Or, you might use the following formula: number of blocks $= \frac{50(50 + 1)}{2}$, or 1275.

(f) number of blocks $= \frac{s(s + 1)}{2}$

QUESTION 8 Page 5

Students can enter the values for the number of steps in one list and the corresponding number of blocks in another list. After they create the graph, ask them to describe its shape. Remind students that, in *Constructing Mathematics, Book 1,* they looked at similar graphs when they studied quadratics.

Answers



8. (a) See the scatter plot in the margin, made by a graphing calculator.

(b) *example:* Its shape is not linear; the y-values change more as the x-values increase; its shape seems to be like the graph of a quadratic relation; that is, part of a parabola.

(c) A non-linear relationship exists because (1) the graph is not a straight line, and (2) the rate of change is not constant. For example, between the points (1, 1) and (2, 3), the rate of change is $\frac{3 - 1}{2 - 1} = 2$, but between the points (2, 3) and (3, 6), the rate of change is $\frac{6 - 3}{3 - 2} = 3$.

QUESTIONS 9, 10, AND 11 Page 5

Answers

9. (a) *example:* Divide a term by 2 to get the next term: {80, 40, 20, 10, 5, 2.5, 1.25, …}.

(b) *example:* Each number is a cube: $\{1^3, 2^3, 3^3, 4^3, 5^3, 6^3, 7^3, 8^3, \ldots\}$.

(c) *example:* Each term is a person's name, the names are in alphabetical order, and each name begins with the appropriate letter of the alphabet.
{Alvin, Barbara, Carla, Dennis, Evelyn, Fred, Gary, …}

(d) *example:* Each term is a fraction formed by increasing both the numerator and denominator by 1: $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \ldots\right\}$.

(e) *example:* Each term is five times the value of the previous term: {6, 30, 150, 750, 3750, 18 750, 93 750, …}.

10. (a) The ellipsis marks a pattern that repeats indefinitely.
    (b) All are infinite except for part (c), which will likely end with a name that starts with the letter Z. The sequence might also continue with the letter A and repeat indefinitely.
11. Answers will vary.

FOCUS A

Using Patterns to Solve Problems

[Suggested time: 60 min] ***[Text page 6]***

Students use the problem-solving strategies of solving a simpler problem, drawing diagrams, and completing sequences or tables. They identify number patterns in the data of the sequences or tables and extend the patterns to solve the given problem.

Have students read the staircase problem in the Example, and make sure that they see the difference between a step and a jump. Have students place the letters S and J (for a step and a jump) in the squares of grid paper. For example, the diagram shows the ways to climb four stairs.

<table>
<tr><td>S</td><td>S</td><td>S</td><td>S</td></tr>
<tr><td>S</td><td>S</td><td colspan="2">J</td></tr>
<tr><td>S</td><td colspan="2">J</td><td>S</td></tr>
<tr><td colspan="2">J</td><td>S</td><td>S</td></tr>
<tr><td colspan="2">J</td><td colspan="2">J</td></tr>
</table>

Some students might see that this problem is the same as finding sums using the addends 1 and 2 in any order.

1 = 1	1 way
2 = 1 + 1, 2	2 ways
3 = 1 + 1 + 1, 1 + 2, 2 + 1	3 ways
4 = 1 + 1 + 1 + 1, 2 + 1 + 1, 1 + 2 + 1, 1 + 1 + 2, 2 + 2	5 ways

Focus Questions

QUESTIONS 12, 13, AND 14 Page 7

Students will likely need guidance to find the number of ways to climb four and then five stairs systematically and not randomly. Students who use guess and check only will likely not find all possible pairs and, therefore, not see a pattern. For example, to find the number of ways to climb five stairs, suggest that they first list a way using only steps:

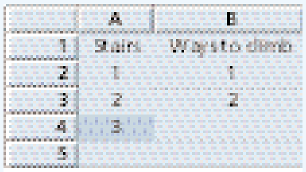
S-S-S-S-S	
S-S-S-**J**	Keep the first step constant and replace pairs of steps with a jump.
S-S-**J**-S	
S-**J**-S-S	
J-S-S-S	The remaining ways must involve combinations of jumps and a step.
J-**J**-S	
J-S-**J**	
S-**J**-**J**	

Management Tip

Most people would likely predict that there are four different ways to climb four stairs, because the pattern appears to be consecutive whole numbers: 1, 2, 3, and then 4. Use this example to show that more data should be gathered to check any prediction. In this case, you would make an organized list or a drawing to confirm or disprove the prediction of four ways.

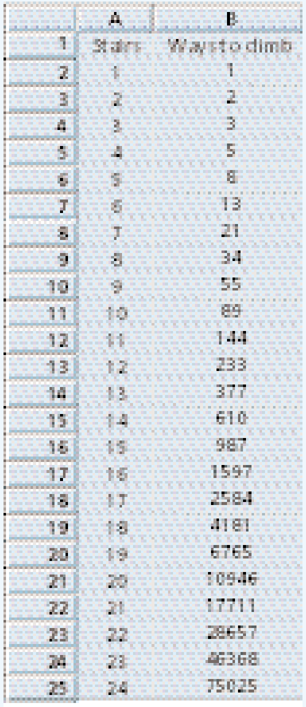
Challenge Yourself

If you use a staircase in a local building, students might need a graphing calculator or spreadsheet to do the large number of calculations. For example, suppose that a student's two-storey home has 24 steps. In the first diagram, the numbers "1" and "2" (representing the number of ways to climb one and two steps) are entered as solutions in cells B2 and B3. The spreadsheet is told to place the sum in cell B4. Highlighting cells A4 and B4 and then dragging the corner of cell B4 completes the table automatically.



	A	B
1	Stairs	Ways to climb
2	1	1
3	2	2
4	3	
5		

For a staircase with 24 steps, the spreadsheet might look like this:



	A	B
1	Stairs	Ways to climb
2	1	1
3	2	2
4	3	3
5	4	5
6	5	8
7	6	13
8	7	21
9	8	34
10	9	55
11	10	89
12	11	144
13	12	233
14	13	377
15	14	610
16	15	987
17	16	1597
18	17	2584
19	18	4181
20	19	6765
21	20	10946
22	21	17711
23	22	28657
24	23	46368
25	24	75025

There would be more than 75 000 ways to climb the stairs in a two-storey house using combinations of *steps* and *jumps* only.

Students could role-play climbing on a staircase.

Have students prepare a master list on the board so that they can see if any ways of climbing have been missed or duplicated. Once students have found the numbers of ways to climb from one to five stairs, they can use the table or sequence to name the pattern and extend it to find the number of ways to climb 10 stairs.

Number of stairs	1	2	3	4	5	6	7	8	9	10
Number of ways to climb stairs	1	2	3	5	8	13	21	34	55	89

Note that the pattern is *recursive*, and that each term is the sum of the previous pair of terms. The sequence, {1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …}, is very much like the Fibonacci sequence that will be presented in Investigation 2. Students do not need to write or use the term "recursive." It is used here as some students may wish to define the type of pattern used. Students will work with the idea, not the terminology. (Later, the phrase "Fibonacci-like" is used informally to describe some sequences that can be defined recursively.)

Check Your Understanding

QUESTIONS 15 AND 16 Page 7

Some students might give impulsive answers such as "the number of handshakes is the square of the number of students." To discourage this, have students shake hands in groups of four or five to find the number of handshakes for two, three, four, and five people. These results should be written in a table on chart paper or on the board.

A chart such as the following makes it clear why adding one person to a group increases the number of handshakes in the group by the original number of people in the group. Students should see a pattern that each new number of handshakes is the sum of the two previous numbers of handshakes, such as 1 + 2 + 3 + 4 = 10 handshakes for five students.

	Jack	Kim	Colin	Kevin	Marian
Jack		✔	✔	✔	✔
Kim			✔	✔	✔
Colin				✔	✔
Kevin					✔
Marian					

Once the problem is solved for two to five students, have the class put the collected data into a table as shown.

Number of students	2	3	4	5	6	7	8	9	10	11
Number of handshakes	1	3	6	10						

Although differences between adjacent terms are not introduced formally until Investigation 3, most students can be expected to use differences to complete the table as shown.

Number of students	2	3	4	5	6	7	8	9	10	11
Number of handshakes	1	3	6	10	10 + 5	15 + 6	21 + 7	28 + 8	36 + 9	45 + 10
Differences		2	3	4	5	6	7	8	9	10

This recursive pattern can be used to find the number of handshakes in a class of 12 students by adding 11 to 55 to get 66. The pattern can be extended indefinitely until the total number of handshakes for the students in the class is known.

Other students might find a non-recursive pattern and use it to develop a relationship between the number of students and the number of handshakes.

Number of students	2	3	4	**5**	**6**
Number of handshakes	1	3	6	10	**15**

Have students note that the product of 5 and 6 is 30, which is double the number of handshakes for six students. This appears to suggest that the number of handshakes, h, for n students is one-half of $n(n - 1)$. Thus, the general relation $h = \frac{n(n-1)}{2}$ can be used to solve the problem for any number of people.

Students using a graphing calculator can enter the values for the number of students in one list and the corresponding number of handshakes in another list. After they create the graph, ask them to describe its shape. For part (c), students can use a spreadsheet of differences to find the number of handshakes involving all members in the school. However, it is more practical for them to substitute the number of people in the non-recursive relation $h = \frac{n(n-1)}{2}$.

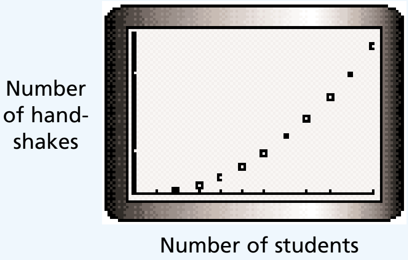
Answers

15. (a) and (b) See the previous discussion.

16. (a) The scatter plot is shown in the margin; the shape is non-linear.
    (b) The graph is not a straight line and, therefore, the relationship is non-linear. Alternatively, the rate of change is not constant, since (for example) $\frac{10-6}{5-4} = 4$ but $\frac{15-10}{6-5} = 5$. There is a linear relationship only if the rate of change is constant, so here the relationship is non-linear.
    (c) *example:* In a school of 485 teachers, staff, and students, the number of handshakes is $h = \frac{485 \times 484}{2} = 117\ 370$.

Management Tip

The chart shows the following relationships: Each student in a class of n students shakes hands with $n - 1$ students. The number of handshakes will be $n(n - 1)$. However, Jack shaking hands with Kim and Kim shaking hands with Jack is considered to be the same handshake. Therefore, we must divide the first expression by 2. The number of handshakes is $\frac{n(n-1)}{2}$.



QUESTION 17 Page 7

Students should be encouraged to notice problems that seem to have the same underlying structure but a different context than previously solved problems. The cables connecting a computer are like a handshake, because only one cable is needed to link two computers and only one handshake is needed to connect two people. In the same way, one line segment is needed to connect two points. Therefore, both problems can be solved using the patterns and generalizations about the data collected in the handshake problem.

Students will likely see the connection between the handshake problem and the ball-tossing problem, but might rightly argue that a ball tossed from Jack to Jill is different from a ball tossed from Jill to Jack.

Answers

17. (a) *example:* lines needed to connect six points in a circle to each other: $5 + 4 + 3 + 2 + 1 = 15$

    lines for six points on a circle: $\frac{(6)(5)}{2} = 15$

    (b) *example:* $10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 55$ cables

    $\text{cables} = \frac{\text{computers}(\text{computers} - 1)}{2}$

    cables for 11 computers $= \frac{(11)(10)}{2} = 55$

    (c) *example:* Tossing a ball between two people can be considered to take two tosses rather than one toss as in the other problems. The number of ball tosses is twice the value obtained in the cable problem, or $2 \times 55 = 110$.

QUESTION 18 Page 8

Have students look at their results from the staircase problem in Question 7 of this section. They should note that the number of blocks needed to build each successive staircase increases by n, the number of steps, and that the number of blocks is the sum of consecutive counting numbers from 1 to n.

$1 \qquad 1 + 2 \qquad 1 + 2 + 3 \qquad 1 + 2 + 3 + 4$ and so on

In the handshake problem, the number of handshakes increases by $n - 1$, the number of students, and the number of handshakes is the sum of consecutive counting numbers from 1 to $n - 1$.

$1 \qquad 1 + 2 \qquad 1 + 2 + 3$ and so on

Both problems can be solved using these relationships: $\frac{n(n-1)}{2}$ = number of handshakes between n people = number of blocks to make $n - 1$ steps.

Answer

18. See the previous discussion.

Investigation 2

Properties of a Special Sequence

[Suggested time: 60 min] ***[Text page 8]***

Purpose

Define and explore some properties of the Fibonacci sequence.

Management Suggestions and Materials

Students need to understand that

- pairs rather than individual rabbits are being considered.
- adult rabbits can reproduce during the second month and thereafter. Baby rabbits have to wait an additional month before reproducing.

Students might use pairs of white cube-a-links for pairs of baby rabbits, and pairs of brown cube-a-links for pairs of adult rabbits.

Ensure that students understand the diagram and the organized list in the introductory rabbit situation.

Procedure

Step A

Most students will likely prefer to make an organized list using letters or numbers rather than sketches of rabbits to show the family tree of Fibonacci's rabbits. Each pair of letters is one pair of rabbits: AA is a pair of adult rabbits, and BB is a pair of baby rabbits. The list on page 9 of the Student Book can be extended to give the results below. Expect some variation in the order in which pairs of adult and baby rabbits are listed.

Start of Month 1	Start of Month 2	Start of Month 3	Start of Month 4	Start of Month 5
BB	**AA**	**AA, BB**	**BB, AA, AA**	**AA, BB, AA, AA, BB**
1 pair	1 pair	2 pairs	3 pairs	5 pairs

This pattern can be continued as follows:

Start of Month 6	Start of Month 7	Start of Month 8
BB, AA, AA, BB, AA, AA, BB, AA	**AA, BB, AA, BB, AA, AA, BB, AA, AA, BB, AA, AA, BB**	**BB, AA, AA, BB, AA, AA, BB, AA, AA, BB, AA, BB, AA, AA, BB, AA, BB, AA, AA, BB, AA**
8 pairs	13 pairs	21 pairs

Steps B, C, and D

The results of an organized list or a family tree found in Step A can be made into a sequence: {1, 1, 2, 3, 5, 8, 13, 21, …}. By continuing the obvious pattern that a term is the sum of the two previous terms, the 13 terms representing one year of reproduction are {1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233}. Thus, there will be 233 pairs of rabbits when the original pair turns one year old, at the start of month 13.

Think about …

Steps B and C

Students should see that the patterns are similar, except that the staircase sequence has 1 and 2 as its two initial terms, while the Fibonacci sequence has the two initial terms 1 and 1. Both sequences come from the rule *Add the previous two terms to create the next term.*

Notebook Entry

Have students copy the definition and examples of a Fibonacci sequence and number into their notebooks.

Technology

Students who want to know the value of the 50th Fibonacci number might use a spreadsheet, graphing calculator, or the Internet to produce a list of at least 50 Fibonacci numbers. For example, this site presently contains the first 500 Fibonacci numbers, in blocks of 100: **math.holycross.edu/~davids/fibonacci/fibonacci.html**

Note

For Question 21, there are non-recursive methods that are beyond the scope of the course.

Note

Students should realize that some flowers in the table, such as the iris and buttercup, virtually always display the same number of petals. Other plants have a tendency to have petals that are close to a Fibonacci number. On average, however, the plants listed in the table have petals whose numbers are Fibonacci numbers.

The Chapter Project at the end of this section lets students further explore some examples of the Fibonacci sequence found in nature.

Investigation Questions

QUESTIONS 19 AND 20 Page 9

Make sure that students choose *consecutive* Fibonacci numbers in parts (a) and (b).

Answers

19. (a) The third number is the sum of the previous two Fibonacci numbers.
    (b) *example:* Suppose the first 10 Fibonacci numbers have been removed. The remaining numbers form a Fibonacci-like sequence, where each term is the sum of the two previous terms: {89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, ...}.
20. Because the 19th and 20th Fibonacci numbers are consecutive,
    (a) The sum 4181 + 6765 is 10 946, which is the 21st Fibonacci number.
    (b) The difference between the 20th and the 19th Fibonacci numbers will be the 18th Fibonacci number: $6765 - 4181 = 2584$.

QUESTION 21 Page 10

Students will see that this is a recursive procedure and that they can find the 50th Fibonacci number only by continuing to find the sum of consecutive terms.

Answer

21. Continue to apply the rule, adding the two previous Fibonacci numbers to get the next, until the 50th number is reached.

QUESTION 22 Page 10

Have students do some research on the flowers of the plants listed in the table. They can get information from gardening books, encyclopedias, local florists, and so on. Using the plant names as keywords in an Internet search will turn up gardening suppliers who usually include photographs of flowers.

Answers

22. (a) *example:* Each term is the sum of the previous two terms; each term is a Fibonacci number.
    (b)

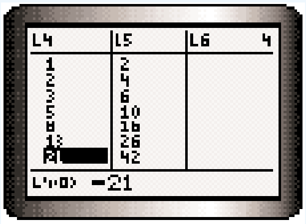
Petal count	**Plants**
2	enchanter's nightshade
3	lily, iris
5	buttercup, wild rose
8	delphinium, bloodroot
13	ragwort, mayweed
21	black-eyed Susan, chicory, aster
34	field daisy, plantain
55	African daisy, helenium
89	Michaelmas daisy

Check Your Understanding

[Completion and discussion: 60 min]

QUESTION 23 Page 10

Suggest that students enter the first 15 Fibonacci numbers into a LIST editor of a graphing calculator or the cells of a spreadsheet. In this way, calculations will be more accurate and patterns become more evident. In the calculator window in the margin, List L4 has the original Fibonacci sequence, and List L5 has the Fibonacci number doubled. Clearly, the transformed sequence works in the same way as the original Fibonacci; that is, each term is the sum of the two previous terms.



```
L4      |L5      |L6     4
   1    | 2      |
   2    | 4      |
   3    | 6      |
   5    | 10     |
   8    | 16     |
   13   | 26     |
   21   | 42     |
L4(7) =21
```

Answers

23. (a) {2, 2, 4, 6, 10, 16, 26, 42, 68, 110, 178, 288, 466, 754, 1220}
    (b) {6, 6, 7, 8, 10, 13, 18, 26, 39, 60, 94, 149, 238, 382, 615}
    (c) {0.5, 0.5, 1, 1.5, 2.5, 4, 6.5, 10.5, 17, 27.5, 44.5, 72, 116.5, 188.5, 305}
    (d) Multiplying and dividing by a constant as in parts (a) and (c) will change a Fibonacci sequence into a sequence with similar properties, but adding or subtracting a constant as in part (b) will not.

QUESTION 24 Page 10

Have a student create a spreadsheet or use a graphing calculator and report the results to the class.

Fibonacci number	1	1	2	3	5	8	13	21	34	55	89
Quotient	1	0.5	0.666 667	0.6	0.625	0.615 385	0.619 048	0.617 647	0.618 182	0.617 978	0.618 056

Answers

24. (a), (b), and (c) The greater the pair of adjacent Fibonacci numbers, the closer the quotient gets to 0.618.

Management Tip

To shorten the number of calculations in Questions 24 and 26, assign different quotients to various students. The results can then be placed into a class table for all students to see and record in their notebooks.

QUESTION 25 Page 11

Mention that about 100 years ago, the German psychologist Gustav Fechner measured the dimensions of thousands of common rectangles and found that, on average, the ratio of the shorter side to the longer side of these rectangles was close to the golden ratio, or about 0.618. He also tested people's preferences, and found that rectangles in the proportions of the golden rectangle were preferred over other proportions. Most rectangular labels, ads, packages, and displays take advantage of this preference.

Suggest that students measure index cards, playing cards, writing pads, windows, postcards, light switch plates, credit cards, bank cards, and so forth. The dimensions can be recorded in a class chart. Generally speaking, the average ratio should be close to the golden ratio. For example, a 3 by 5 and a 5 by 8 index card have ratios of 0.6 and 0.625, respectively. These ratios result no matter what measurement units (inches or centimetres) are used.

Note

Interested students should be told that the expression $\frac{\sqrt{5}-1}{2}$ can be used to find the golden ratio, although the reason why is beyond the level of these students. For example, the expression taken to eight decimal places is $\frac{\sqrt{5}-1}{2} \doteq 0.618\ 033\ 99$. The value 0.618 is close enough for all practical purposes.

Answers

25. (a) *example:* 10 cm by 16 cm results in a ratio of 0.625.
    (b) *example:* about 1.618
    (c) Answers will vary. For example, picture frames, post cards, and so on.

QUESTION 26 Page 11

Have a student create a spreadsheet or use a graphing calculator and report the results to the class.

Fibonacci number	1	1	2	3	5	8	13	21	34	55	89
Quotient	1	2	1.5	1.666 667	1.6	1.625	1.615 385	1.619 048	1.617 647	1.618 182	1.617 978

Answers

26. (a), (b), and (c) The greater the pair of adjacent Fibonacci numbers, the closer the quotient gets to 1.618.

QUESTION 27 Page 11

Have students read Question 27 while you distribute **Blackline Master 1.1.1**. Ensure that students have rulers or measuring tapes calibrated in millimetres. Have them record the line segments that differ in length and their measurements in a table such as the following:

Line segment	$\overline{AB}$	$\overline{FA}$	$\overline{FJ}$	$\overline{FI}$
Length (mm)	34	55	89	144

Some students might note that the number of holes in the sand dollar is also a Fibonacci number.

Answer

27. *example:* The numbers in the measurements are Fibonacci numbers.

QUESTION 28 Page 12

Answers

28. (a) 1 + 1 = 2
1 + 1 + 2 = 4
1 + 1 + 2 + 3 = 7
1 + 1 + 2 + 3 + 5 = 12
1 + 1 + 2 + 3 + 5 + 8 = 20
1 + 1 + 2 + 3 + 5 + 8 + 13 = 33

(b) {2, 4, 7, 12, 20, 33}

{4 − 2, 7 − 4, 12 − 7, 20 − 12, 33 − 20}

{2, 3, 5, 8, 13, …}

(c) All differences are Fibonacci numbers.

QUESTION 29 Page 12

Answers

29. (a) $1^2 = 1 \times 1$

$1^2 + 1^2 = 1 \times 2$

$1^2 + 1^2 + 2^2 = 2 \times 3$

$1^2 + 1^2 + 2^2 + 3^2 = 3 \times 5$

$1^2 + 1^2 + 2^2 + 3^2 + 5^2 = 5 \times 8$

$1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 = 8 \times 13$

(b) The sum of the squares of the first n consecutive Fibonacci numbers is the product of the n^{th} and $(n + 1)^{\text{th}}$ Fibonacci numbers.

QUESTION 30 Page 12

Give students **Blackline Master 1.1.2**, which contains several lists of the first 40 Fibonacci numbers. Students can then easily circle or shade every third, fourth, fifth, and sixth Fibonacci number to test the statements, which are all true. The table shows that every third Fibonacci number is divisible by 2.

1	1	**2** **Divisible by 2**	3	5	**8** **Divisible by 2**
13	21	**34** **Divisible by 2**	55	89	**144** **Divisible by 2**
233	377	**610** **Divisible by 2**	987	1 597	**2 584** **Divisible by 2**
4 181	6 765	**10 946** **Divisible by 2**	17 711	28 657	**46 368** **Divisible by 2**
75 025	121 393	**196 418** **Divisible by 2**	317 811	514 229	**832 040** **Divisible by 2**
1 346 269	2 178 309	**3 524 578** **Divisible by 2**	5 702 887	9 227 465	**14 930 352** **Divisible by 2**
24 157 817	39 088 169	**63 245 986** **Divisible by 2**	102 334 155	…	…

Answers

30. (a) to (d) are true.

QUESTION 31 Page 12

When students have finished Question 30, they should have noted that each statement is true. Therefore, they simply have to extend the generalization to other numbers. Have them show that their predictions are valid by listing the relevant Fibonacci numbers.

Again, copies of **Blackline Master 1.1.2** will be useful. The following table shows that every seventh Fibonacci number is divisible by 13.

1	1	2	3	5	8	13 **Divisible by 13**
21	34	55	89	144	233	377 **Divisible by 13**
610	987	1 597	2 584	4 181	6 765	10 946 **Divisible by 13**
17 711	28 657	46 368	75 025	121 393	196 418	317 811 **Divisible by 13**
514 229	832 040	1 346 269	2 178 309	3 524 578	5 702 887	9 227 465 **Divisible by 13**
14 930 352	24 157 817	39 088 169	63 245 986	102 334 155	…	…

Answer

31. *examples:* Every seventh Fibonacci number is divisible by 13.
Every eighth Fibonacci number is divisible by 21.

QUESTION 32 Page 12

After students create the Lucas sequence, have them see which relationships in the Check Your Understanding questions are true for Lucas numbers. For example, ask some groups to find the effect of adding, subtracting, multiplying, or dividing each term by a constant; others might want to remove consecutive terms; others might want to try the patterns in Question 28 and 29. Still others might want to see if the divisibility rules in Question 30 apply to Lucas numbers.

Answer

32. {1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322}

Challenge Yourself

In this exploration, students are able to explore Fibonacci-like sequences applying the rule *Add the previous two terms to get the next.* You might want to use the activity to explore spreadsheets or programs written for the TI-83 graphing calculator.

Students will likely note that because 2 and 3 are Fibonacci numbers, the remaining numbers are also Fibonacci numbers. Students will find that a Fibonacci-like sequence will retain the following rule: *Add the previous two terms to get the next even when any number of consecutive terms are eliminated from the sequence.* They will also find that multiplying and dividing each term by a constant will produce a Fibonacci-like sequence.

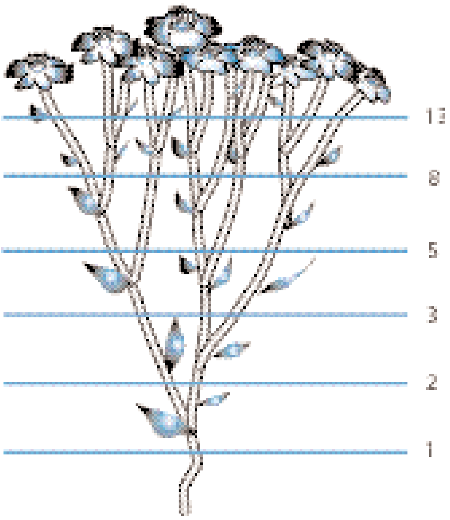
Have students share the tasks of exploring the properties of the Fibonacci-like sequence. For example, different groups might look at the sequence as it applies to questions in the Check Your Understanding section.

Plant and Tree Growth

In this project, students explore the growth of trees and plants under ideal conditions. You might mention that no two trees grow exactly alike, and in any given tree, some branches are thicker and some have odd shapes and wiggles. The way a tree grows and branches is partly due to its nature, but mostly due to its interaction with the environment. However, plant growth does show some patterns similar to those studied in this chapter.

Ensure that you discuss students' answers before completing Section 1.2 and assigning the next part of the project.

(a) and (b) The diagram and sequence showing the number of branches in the upper parts of the tree are shown.



The sequence is {1, 2, 3, 5, 8, 13, …}, which is similar to the Fibonacci sequence with the exception of the initial two terms.

(c) Students can guess that, if the plant continued to branch, the next set would have 8 + 13 = 21 branches.

(d) Have students make a chart for the number of branches. A letter refers to one particular branch and the subscript tells the age of that branch.

Month	**Branches and ages**	**Number of branches**
1	A_1	1
2	A_2	1
3	A_3-B_1	2
4	A_4-C_1, B_2	3
5	A_5-D_1, B_3-E_1, C_2	5
6	A_6-F_1, B_4-G_1, C_3-H_1, D_2 E_2	8

Most students will see the similarity between the plant growth and the rabbit reproduction problem from Investigation 2.

(e) Students should note that the sequence in part (d) is a Fibonacci sequence with two initial terms of 1 instead of only one initial term of 1 as in the sequence in part (a).

(f) When students research and report on other properties of Fibonacci numbers found in nature, they can use keywords such as "Fibonacci" in an Internet search. One very useful site is mentioned in the Did You Know?: **www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/fibnat.html**

A good book on this topic is *Fascinating Fibonaccis: Mystery and Magic in Numbers* by Trudi Hammel Garland, published in 1987 by Dale Seymour Publications.

The following is a brief list of some examples of Fibonacci numbers found in nature:

- the family tree of a male drone bee
- the Fibonacci Spiral and the shape of a snail shell or some sea shells
- the arrangement of seeds on flowerheads
- the arrangements of leaves around their stems
- the arrangements of florets on cauliflower and broccoli
- the flat surfaces enclosing a banana
- the seed pocket of an apple

1.2 Number Patterns: Part 1

Suggested instruction time: 2 hours

Purpose of the Section

Students will look at properties of arithmetic sequences (sequences with constant or regular growth). They will solve arithmetic sequence problems by first identifying patterns and then extending them. They will also translate words into a rule for building an arithmetic sequence. They will graph the term number and term value of arithmetic sequences, and use the graph to make a rule for building the sequence. They will classify arithmetic sequences by making a table or sequence of differences between terms, and use the first-level common difference, D_1, to find a linear relation that can be used to build a given arithmetic sequence. They also learn more about the relationship between the slope of the graph of the terms of an arithmetic sequence and the common difference, D_1.

CURRICULUM OUTCOMES (SCOs)	RELATED ACTIVITIES	STUDENT BOOK
■ demonstrate an understanding of patterns that are arithmetic, power, and geometric C4	■ solve problems by identifying and extending number patterns in arithmetic sequences	p. 14
■ relate arithmetic patterns to linear relations C7	■ explore graphs of arithmetic sequences	p. 15
	■ use sequences of differences and graphing to identify arithmetic sequences	p. 15
	■ use D_1 (the first-level common difference), words, and sequence notation to find the relationship that builds an arithmetic sequence	p. 17
	■ use graphs of arithmetic sequences to find the relationship that builds an arithmetic sequence	p. 17
	■ relate the slope of the graph of an arithmetic sequence to the common difference, D_1	p. 17