ASSUMED PRIOR KNOWLEDGE
■ definitions and distinguishing characteristics of linear relations ■ using symbols to represent linear relations ■ calculating slope and y-intercept of a line and using this information to develop a linear relation ■ using technology to graph linear relations

NEW TERMS AND CONCEPTS	PAGE
■ sequence of differences	15
■ common difference between successive terms in a sequence	15
■ arithmetic sequence	15
■ first-level sequence of differences, D_1	18
■ t_n, the n^{th} term of a sequence	19

Investigation 3

Patterns of Growth

[Suggested time: 60 min] ***[Text page 14]***

Purpose

Students will explore arithmetic sequences, and find that when they subtract successive terms of a sequence, the answer is always the same constant number. For example, if t_n represents the terms of a sequence, students will find that $t_5 - t_4 = t_4 - t_3 = t_3 - t_2$, and so on, recognizing that the result of the subtraction is the same each time.

The differences between successive terms make another sequence, called the sequence of differences. Students will use this constant number, the first-level common difference, D_1, and the value of the first term to come up with a rule or linear relation to describe any term of the sequence.

They will also build on their knowledge of linear relations (introduced in *Constructing Mathematics, Book 1*) by finding the slope and y-intercept of a line, and using them to make a rule or relation for a given sequence. They learn that the common difference, D_1, of a sequence and the slope of the line on the graph of the sequence are equivalent.

Management Suggestions and Materials

Students need grid paper or graphing calculators to make the graphs. They will also use grid paper to make drawings of fences. To draw the equilateral triangle and hexagon shapes, you might have them trace around pattern blocks or use isometric dot paper.

Procedure

Steps A to C

Students can draw adjoining squares on grid paper to show sections of fencing. Make sure that students write {4, 7, 10, 13, 16, 19} for the first six terms of the sequence.

Note

Students would likely have seen similar problems in the context of joining cube-a-links in *Constructing Mathematics, Book 1*.

Think about ...

Step D

Introduce the sequence notation and make sure that, in the sequence {4, 7, 10, 13, 16, 19}, students know that $t_1 = 4$, $t_2 = 7$, and $t_3 = 10$.

Notebook Entry

Have students copy the definitions and terms related to sequences, with examples to illustrate the ideas.

Steps D and E

Students should notice that, although each section of fencing uses four metal rods, the total number of rods in the fence increases by only 3 when another section is added, because one rod acts as a common side for two adjacent sections.

Some students will likely use a recursive pattern to solve the problem, repeatedly adding 3 to make a sequence with 63 terms. They should understand that extending a recursive pattern to solve a sequence problem is practical only when there are few terms. Non-recursive patterns are more useful when there are many terms.

A non-recursive pattern can be found by noting that the 10th term equals 31, and the rule for building the sequence might be *Multiply 3 by the term number and add 1*. If this is true, a 63-section fence will use $3 \times 63 + 1 = 190$ rods. Other terms can be compared to confirm this idea.

Students might come up with a non-recursive pattern by thinking something like this:

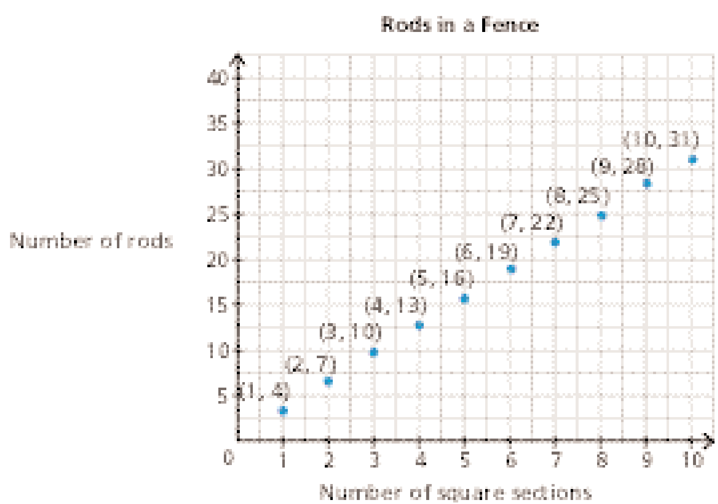
> Because the first section of the fence has four metal rods and because the remaining 62 sections need $62 \times 3 = 186$ metal rods, the total number of rods is $186 + 4 = 190$.

This reasoning can be followed to a more general rule that will be developed more in Steps F to H and treated more formally in Investigation 4.

> The number of rods in the first section of the fence (or t_1) is 4. The remaining $(n - 1)$ terms are multiples of 3. Therefore, term $t_n = 4 + (n - 1)3$, or $t_n = 4 + 3n - 3 = 3n + 1$. This general rule can be seen to generate the sequence when 1, 2, 3, 4, … are substituted for n.

Steps F to H

When students graph the number of rods in a fence versus the number of square sections, as shown in the following graph, help them to find the slope and the y-intercept of the line. These calculations were introduced in *Constructing Mathematics, Book 1,* and should have been reviewed in Section 1.1.



The slope is easy to find because the x-values are consecutive term numbers of a sequence and, so, differ by only 1. Therefore, the run or horizontal difference between consecutive points will equal 1.

$$\text{slope} = \frac{\text{amount of rise}}{\text{amount of run}} = \frac{7 - 4}{1} = 3$$

Have students pick non-consecutive points to show that the slope is always 3.

$$\text{slope} = \frac{\text{amount of rise}}{\text{amount of run}} = \frac{31 - 4}{10 - 1} = \frac{27}{9} = 3$$

Remind students that the general form of a linear relation was introduced in *Constructing Mathematics, Book 1*. Known as the *slope–y-intercept form*, it is written as $y = b + mx$, where m is the slope and b is the y-intercept. The rule for building an arithmetic sequence can be expressed like this as well.

y-intercept—the point where a graph crosses the y-axis; this is the point where $x = 0$

Students should realize that the slope of the graph shows the constant growth from one term to another. In this case, the constant growth is 3 (or three rods, using the fence context). The first fence had four rods, and the initial term in the sequence is 4. So, the rule in words is *Add multiples of 3 to 4.* Because the first multiple of 3 added to 4 is 3, students will have to see that, in general, $n - 1$ multiples of 3 are added where n is the term number. This relationship can be expressed in symbols using sequence notation as $t_n = 4 + 3(n - 1)$ or $t_n = 3n + 1$. Therefore, the y-intercept is 1.

These rules or patterns can be used to find the number of metal rods in a 63-section fence, which equals $63 \times 3 + 1 = 190$.

Investigation Questions

QUESTION 1 Page 15

In doing part (a), encourage students to collect data systematically and write the results in a table or as a sequence. For Shape 1, note that the sequence begins with a 5 because the first section uses five rods. Thereafter, each term increases by 4 because adjacent fence sections share a common rod. For Shape 2, the first term is a 6 but, thereafter, the number of rods increases by 5.

Shape 1 sequence: {5, 9, 13, 17, 21, 25, 29, 33, 37, 41, …}
Shape 2 sequence: {6, 11, 16, 21, 26, 31, 36, 41, 46, 51, …}

Each sequence can be extended to find the total number of rods in a 63-section fence. The Shape 1 sequence can be extended by repeatedly adding 4, while the Shape 2 fence can be extended by repeatedly adding 5.

Students may note, for example, that the single-diagonal fence (Shape 1) has a first square with five rods and 62 squares with four rods, for a total of $5 + 62 \times 4 = 253$ rods. The double-diagonal fence (Shape 2) has a first square with six rods and 62 squares with five rods, for a total of $6 + 62 \times 5 = 316$ rods. These observations can be developed into two relations:

For Shape 1, the relation is *Count the number of rods in the first section (5). Count the number of sections added to the first section and multiply this number by 4. Add 5 to the number of rods in the added sections*. This relation can become a rule using sequence notation: $t_n = 5 + 4(n - 1)$, or $t_n = 4n + 1$.

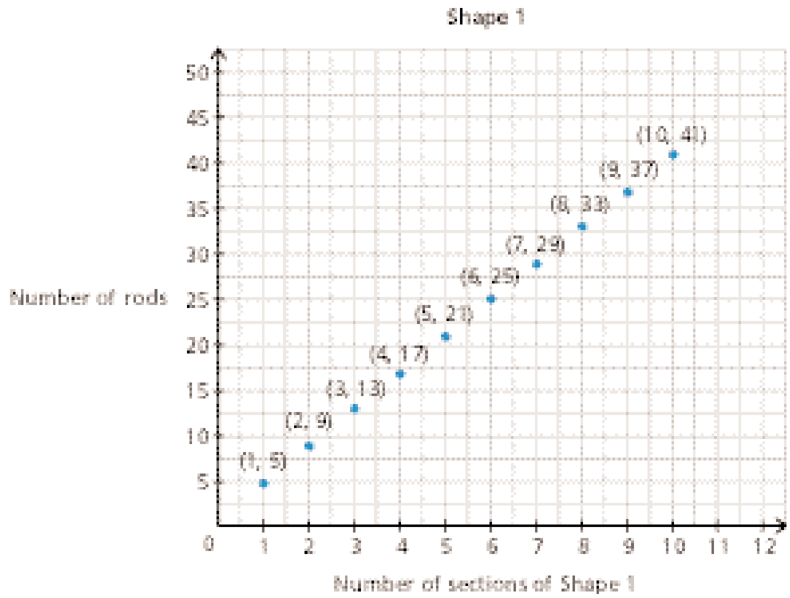
For Shape 2, the relation is *Count the number of rods in the first section (6). Count the number of sections added to the first section and multiply this number by 5. Add 6 to the number of rods in the added sections.* This relation can be developed into a rule using sequence notation: $t_n = 6 + 5(n - 1)$, or $t_n = 5n + 1$.

Management Tip

Students at this level should be encouraged to express a rule or relationship with numbers first and then symbols. For example, if they understand that the number of rods needed to build a 63-section fence with Shape 1 is $5 + 62 \times 4$, or $5 + (63 - 1) \times 4$, they can replace $63 - 1$ with the more general expression $n - 1$. Therefore, the general rule for finding the number of rods needed to build a fence with n sections of Shape 1 is $t_n = 5 + (n - 1) \times 4$.

Also have students graph the number of rods in a fence versus the number of sections made from each shape. Students may draw each graph on graph paper or use graphing technology. Ensure that they choose a suitable scale for each graph. Encourage students to explain why the graphs do not intersect the origin, and why the graph of each sequence lies on a straight line. They might want to connect the points on the graphs but should not, as the domain of each graph is only the positive integers.

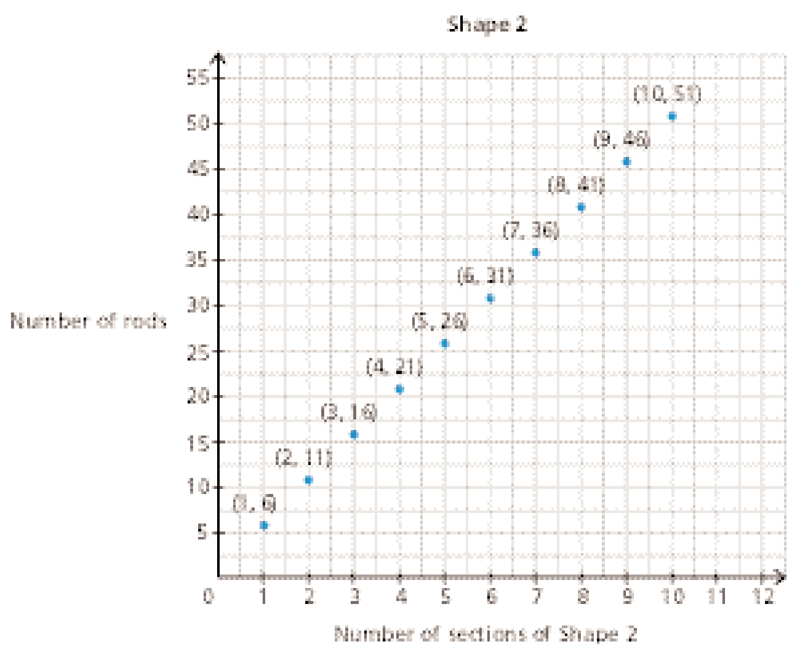
The slope of the graph measures the constant increase from one term to the next. In the case of Shape 1, the slope or constant increase is 4.



$$\text{slope} = \frac{\text{amount of rise}}{\text{amount of run}} = \frac{9 - 5}{1} = 4$$

Because the slope or constant increase is 4, a 63-section fence of Shape 1 contains $63 \times 4 + 1 = 253$ rods.

In the case of Shape 2, the slope or constant increase is 5.



Think about ...

The Sequences in Question 1

Each sequence is formed by the simple "arithmetic" of adding the same whole number to each term. Remind students that adding a negative integer is the same as subtracting a positive number. Therefore, adding any real number can make an arithmetic sequence. Later in this Investigation, students will look at sequences made by adding any real number.

Notebook Entry

After students do the Think about..., have them write their definition of an arithmetic sequence, along with some examples that show addition of positive and negative numbers. The term "arithmetic sequence" is used throughout the rest of Section 1.2. An arithmetic sequence is a sequence whose terms differ by a constant number.

$$\text{slope} = \frac{\text{amount of rise}}{\text{amount of run}} = \frac{11 - 6}{1} = 5$$

Because the slope or constant increase is 5, a 63-section fence of Shape 2 contains $63 \times 5 + 1 = 316$ rods.

Answers

1. (a) Shape 1 sequence: {5, 9, 13, 17, 21, 25}
   Shape 2 sequence: {6, 11, 16, 21, 26, 31}
   (b) Shape 1: $t_n = 4n + 1$
   Shape 2: $t_n = 5n + 1$
   (c) 253 rods are required to construct a 63-section fence for Shape 1;
   316 rods are required to construct a 63-section fence for Shape 2.

QUESTION 2 Page 15

Students should note that because adjacent sections share one common side/rod, any extra section of fencing increases the total number of rods by $n - 1$, where n is the number of sides in the polygon. Encourage them to explain their rules and help them to do so using symbols, especially sequence notation.

Answers

2. (a) *example*: octagon
   (b) *example*: For octagonal sections, the sequence is {8, 15, 22, 29, 36, 43, 50, 57, 64, 71, …}.
   (c) *example*: For octagonal sections, the total number of rods is $8 + 99 \times 7 = 701$. The pattern is to add 7 to each previous number. The rule is $t_n = 8 + (n - 1)7 = 7n + 1$.

QUESTION 3 Page 15

Again, encourage students to explain their rules and any patterns they find.

Answers

3. *Example*: {10, 19, 28, 37, 46, 55, 64, 73, 82, 91, …}
   $19 - 10 = 9$; $28 - 19 = 9$; $64 - 55 = 9$
   Each term is 9 more than the one before.

QUESTION 4 Page 15

Sequences of differences were briefly introduced in Investigation 1. Students should note that each sequence of differences has a common or constant term that is the constant number added to build each arithmetic sequence. So, another definition of an arithmetic sequence is that it is a sequence in which each term in the sequence of differences (later to be called the sequence of first-level differences) is the same number. The notation for the sequence of first-level differences is D_1.

Answers

4. (a) Question 1: Shape 1 {4, 4, 4, ..., 4}
   Shape 2 {5, 5, 5, ... , 5}
   Sequences will vary for Questions 2 and 3.
   (b) Each term is constant.
   (c) *example*: The difference between one term and the next is the same constant number.

sequence of differences—a sequence made from another sequence by subtracting the value of each term in the original sequence from the next term in that sequence, For example, a sequence {1, 3, 5, 7, 9, 11} would be changed to a sequence of differences in this way: $\{3 - 1, 5 - 3, 7 - 5, 9 - 7, 11 - 9\} = \{2, 2, 2, 2, 2\}$.

Note

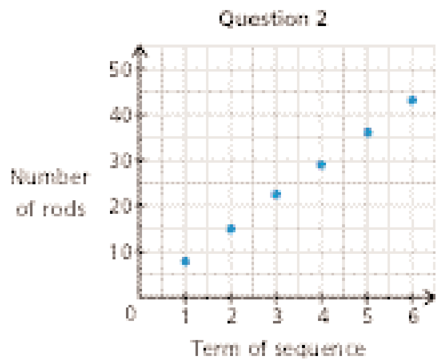
To avoid confusion, have students distinguish between a sequence of differences and the common difference. Subtraction of successive terms in an arithmetic sequence gives a sequence of differences in which each term is a constant number called the *common difference;* for example, {3, 3, 3, ...}.

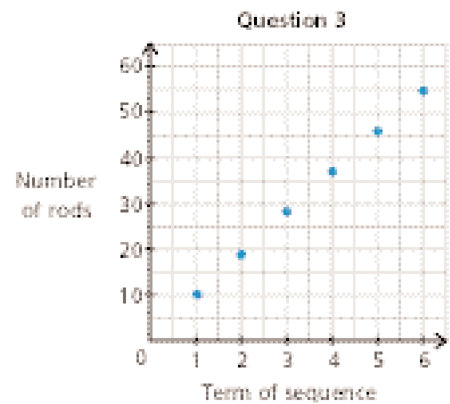
QUESTION 5 Page 15

This question brings up the formal relationship between the slope of an arithmetic sequence and the constant difference, D_1. In working out the slope, students will find that its value is the same as the first-level common difference. Both the slope and D_1 measure the change from one term to the next term, and so they must be the same.

Answers

5. (a) See the discussion of Question 1 for graphs.
examples:





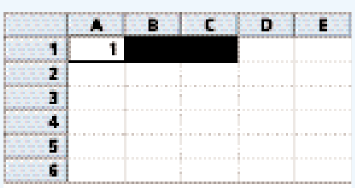
(b) *example*: There is a constant increase from one term to the next.

(c) Slope equals 4 for Shape 1 in Question 1; slope equals 5 for Shape 2 in Question 1; slope equals 7 for the example in Question 2; slope equals 9 for the example in Question 3.

(d) equivalent values

Technology

Show how a spreadsheet can be used to make an arithmetic sequence. Highlighting the initial cells and dragging the lower-right corner automatically gives successive term numbers and the value of each term in the sequence.



QUESTION 6 Page 15

Students should begin to realize that an arithmetic sequence has many properties, including the following, which might be written on chart paper.

Answer

6. Some properties of arithmetic sequences are
   - constant growth
   - a sequence of differences that has a common or constant number
   - the constant number is equal to the slope of the linear relation or rule that builds the arithmetic sequence
   - the graph of the value of a term versus its term number is a straight line

Check Your Understanding

[Completion and discussion: 45 min]

QUESTION 7 Page 16

Students should note that each additional section of triangular railing increases the total number of rods by 2 rather than 3 because adjacent sections share a common side. Also, note that the number of 2s being added is one less than the number of terms; for example, the fourth term has three 2s.

These observations can first be said in words as *Count the number of rods in the first section (3). Count the number of sections added to the first section and multiply this number by 2. Add 3 to the number of rods in the added sections.* This relation can be developed into a rule using sequence notation as $t_n = 3 + (n - 1) \times 2$. This can be simplified to $t_n = 2n + 1$.

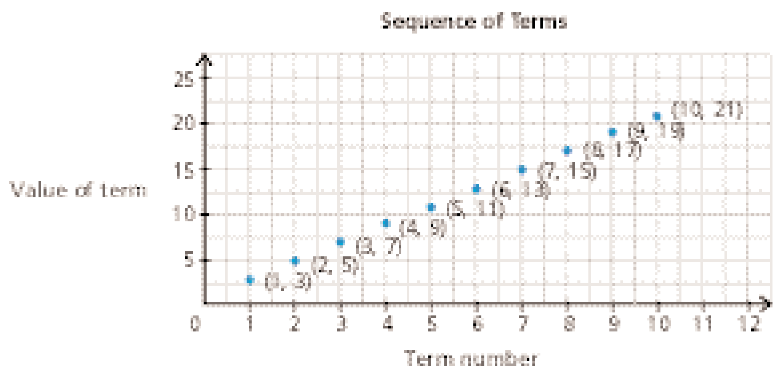
Help some students to develop the rule by using numbers first.

> 200 sections must have $3 + (200 - 1) \times 2$ rods.
> n sections must have $3 + (n - 1) \times 2$ rods.
> $t_n = 3 + 2(n - 1)$

For some students, you might have them ignore the general rule. Instead, expect them to reason with specific numbers. For example, to calculate the number of rods in a 200-section fence, encourage the following thinking:

> *The first section has 3 rods. To make a 200-section fence, 199 sections must be added to the first section and each of these sections contains 2 rods. The total number of rods is $3 + 199 \times 2 = 401$.*

Students should again note that although n, the term number, could be any real number, in practice the domain is only the counting numbers or positive integers.



The graph has a slope and a common difference of 2. So, the rule for the n^{th} term must be $t_n = 3 + 2(n - 1)$, or $t_n = 2n + 1$. Other students may think that each term is one more than double the term number. This will lead them to the relation $t_n = 2n + 1$.

Answers

7. (a) {3, 5, 7, 9, 11, 13, 15, 17, 19, 21}
   (b) See the previous discussion.
   (c) A bridge railing with 200 sections contains $3 + 199 \times 2 = 401$ rods.

QUESTION 8 — Page 16

This question challenges students to use the slope or common difference to make a sequence when the initial term is not given.

In doing part (c), students can find the common difference of 7 by recognizing that $t_4 = 23$ and $t_5 = 30$ and $30 - 23 = 7$.

Note

Questions 7 and 8 are here to develop a general rule for building an arithmetic sequence, given the initial term and the common difference. This is done more formally in Investigation 4.

Students should note that the slope is equal to the common difference:

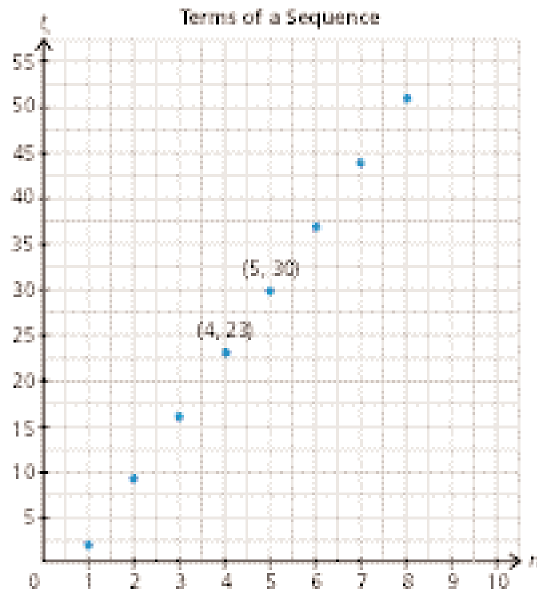
$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{30 - 23}{5 - 4} = 7$$

Students know two terms and the first-level common difference.

{____, ____, ____, 23, 30, ...}
7

Students can use the graph and repeatedly subtract the slope of 7 from the y-value of 23, or repeatedly subtract $D_1 = 7$, to find the other terms of the sequence.

{2, 9, 16, 23, 30, …}
7 7



Answers

8. (a) *example*: The graph is a straight line, showing constant growth from one term to the next.
   (b) 7
   (c) Each equals 7.
   (d) 2, 9, 16
   (e) *example:* Multiply 7 by $n - 1$ and add 2; $t_n = 2 + 7(n - 1)$
   $= 7n - 5$

   Other students may look at the points (1, 2) and (2, 7). Thus, $t_1 = 7n$. Now, if seven ones are seven, then they need to subtract five. Thus, $t_n = 7n - 5$. This approach could be used and verified to get the other terms. This is another way students may think of to find the relation.

   (f) $t_{100} = 7 \times 100 - 5 = 695$

QUESTION 9 Page 17

Students should now have enough knowledge to identify each arithmetic sequence in different ways:

- finding a sequence of differences
- finding the constant number being repeatedly added to the last term of the sequence
- graphing the term number and term value to find out if the graph is a straight line

Answers

9. (a) arithmetic sequence; common difference −4
   (b) arithmetic sequence; common difference 3
   (c) arithmetic sequence; common difference 5

Challenge Yourself

(a) If $D_1 = -5$, start with an initial term and repeatedly subtract 5: {3, −2, −7, −12, −17, −22, …}

(b) If $D_1 = -3.5$, start with an initial term and repeatedly subtract 3.5: {0, −3.5, −7, −10.5, −14, −17.5, …}

(c) If $D_1 = -\frac{3}{4}$, start with an initial term and repeatedly subtract $\frac{3}{4}$: $\left\{100, 99\frac{1}{4}, 98\frac{1}{2}, 97\frac{3}{4}, \ldots\right\}$

(d) If $D_1 = 150$, start with an initial term and repeatedly add 150: {1000, 1150, 1300, 1450, 1600, …}

(d) not an arithmetic sequence; no first-level common difference
(e) arithmetic sequence; common difference 2
(f) not an arithmetic sequence; no first-level common difference
(g) arithmetic sequence; common difference $\frac{1}{4}$
(h) arithmetic sequence; common difference -5

QUESTION 10 Page 17

Encourage students to use different methods of finding the rule or relation used to build the arithmetic sequence. However, do not expect many students to develop a general rule involving symbolism such as in the following reasoning:

(1) The sequence in Question 9(a) is $\{-4, -8, -12, -16, \ldots\}$ and is made by starting with an initial term of -4 and repeatedly adding -4, the slope or first-level common difference. This leads to the rule by taking -4 as the starting number added to $(n - 1)$ multiples of -4. Therefore, $t_n = -4 + (n - 1)(-4) = -4n$.

(2) Because all numbers are multiples of -4, the rule can directly follow as $t_n = -4n$.

Instead, simply have students describe in words how the sequence is created. For example, *Sequence (g) is created by starting with $\frac{1}{2}$ and then constantly adding $\frac{1}{4}$ to each term.*

However, they can use the following reasoning to determine the 200^{th} term in an arithmetic sequence: *The 200^{th} term in sequence (g) will be $\frac{1}{2} + 199 \times \frac{1}{4}$ because 199 quarters are added to the fraction $\frac{1}{2}$.*

Management Tip

The most important idea is to have students describe a relationship with words or symbols, and then use the relation rather than developing a rule using sequence notation.

Answers

10. *example:* Arithmetic sequence (a) created by starting with -4 and then repeatedly adding -4 to each term; 200^{th} term must be $-4 + 199 \times -4$, or -800.

    example: Arithmetic sequence (b) created by starting with 3 and then repeatedly adding 3 to each term; 200^{th} term must be $3 + 199 \times 3$, or 600.

    example: Arithmetic sequence (c) created by starting with 2 and then repeatedly adding 5 to each term; 200^{th} term must be $2 + 199 \times 5$, or 997.

    example: Arithmetic sequence (e) created by starting with 6.5 and then repeatedly adding 2 to each term; 200^{th} term must be $6.5 + 199 \times 2$, or 404.5.

    example: Arithmetic sequence (g) created by starting with $\frac{1}{2}$ and then repeatedly adding $\frac{1}{4}$ to each term; 200^{th} term must be $\frac{1}{2} + 199 \times \frac{1}{4}$, or $50\frac{1}{4}$.

    example: Arithmetic sequence (h) created by starting with 24 and then repeatedly subtracting 5 from each term; 200^{th} term must be $24 - 199 \times 5$, or -971.

QUESTION 11 Page 17

Students can use graphing or sequences of first differences to find that a Fibonacci sequence is not an arithmetic sequence. The simplest difference between the two sequences is that the Fibonacci sequence increases by an increasingly greater number rather than by a constant.

QUESTION 12 Page 17

Students can either form a sequence or realize that the number of rods added to a section is not constant because it is either 3 or 5 depending on what shape has been added.

{4, 9, 12, 17, 20, 25, ...}

5 3

Students can either extend the 5−3 pattern to find the 40th term or think as follows:

> To build a fence with 41 sections, there will be 20 plain squares and 20 squares with diagonal rods attached to the initial square with 4 rods. The total is $4 + 20 \times 3 + 20 \times 5 = 164$ rods.

Answers

12. (a) {4, 9, 12, 17, 20, 25}
    (b) It is not an arithmetic sequence because there is no constant growth from term to term.
    (c) 164 rods

Investigation 4
Generalizing Patterns

[Suggested time: 60 min] ***[Text page 17]***

Think about ...

The Interest and Amount Owed

Students should realize that the sequence of interest {0, 5, 10, 15, 20, 25, 30, ...} and the sequence of the amount owed {100, 105, 110, 115, 120, 125, 130, ...} are each created by adding $5 to the previous month's amount. So, D_1 will be a constant 5 and the sequence is an arithmetic sequence.

Purpose

Students will use a variety of approaches to develop a rule or relation that builds an arithmetic sequence.

Management Suggestions and Materials

Have students read the introductory paragraph to make sure they understand that 5% monthly interest on a sum of $100 will be a constant $5.

Students will need:
- blocks or cube-a-links (optional)
- grid paper

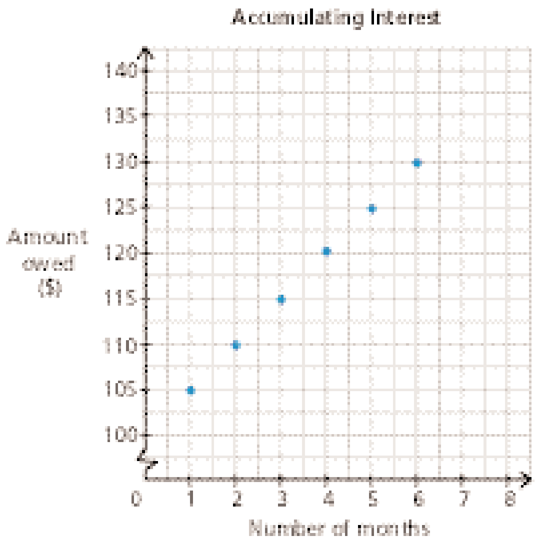
Procedure

Step A

Provide blocks, cube-a-links, or grid paper for students to show the amount of interest for each of the six months. These aids will help students see a visual growth pattern as well as a numerical one.

Steps B to D

Month	0	1	2	3	4	5	6
Interest ($)	0	5	5	5	5	5	5
Total amount owed ($)	100	105	110	115	120	125	130



Students can either use $\frac{\text{rise}}{\text{run}}$ to find the slope, or reason that the constant amount of $5 per month must be the slope.

Investigation Questions

QUESTION 13 Page 18

Students should have had enough experience to know that the slope, the constant growth rate (in this case, the monthly interest), and the first-level common difference are all the same if the number of months (the x-values) increase by increments of one.

Answers

13. (a) $5 (b) equivalent (c) equivalent

QUESTION 14 Page 18

Encourage students to write their relationships with words; for example, *You multiply the number of months by $5 and add this amount to the amount of the loan, $100.*

Then, have them substitute symbols for the phrases:

Multiply the number of months by 5.	$5x$, where x is the number of months
Add this amount to 100.	$y = 100 + 5x$, where y is the amount owed after x months

Again, some students will find it simpler to use an example with numbers, and then replace numbers with symbols.

Amount owed after 6 months	$100 + 6 \times 5$
Amount owed, y, after x months	$y = 5x + 100$

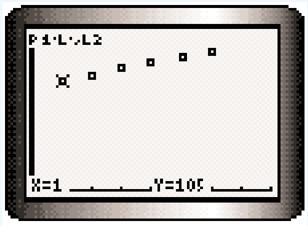
Answers

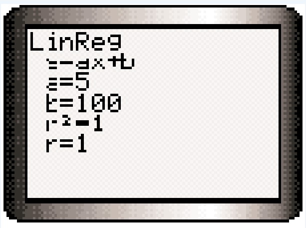
14. (a) *example*: Multiply the number of months by the interest of $5; add the total interest to the amount borrowed, $100.

(b) $y = 5x + 100$, where y is the amount owed after x months

Note

Encourage students to use x and y as variables in Question 14 because, in Question 15, they will find the equation of the curve of best fit.





QUESTION 15 Page 18

The graph shows what students should now expect: there is a linear relationship between the number of months and the total amount owed. Thus, they can use LinReg($ax + b$) to find the equation of the curve of best fit.

The line of best fit is $y = 5x + 100$.

Answers

15. (a) straight line and, therefore, use LinReg($ax + b$)
    (b) $y = 5x + 100$
    (c) the same rule

QUESTION 16 Page 18

Have students discuss how Jason's method can be extended to six months, seven months, and finally n months. Students now have a number of different ways to develop rules for arithmetic sequences.

Answer

16. $t_n = 5n + 100$

QUESTION 17 Page 19

Students can substitute values for n in the sequential rule $t_n = 100 + 5n$ or values of x in the rule $y = 5x + 100$ to find the amount of each debt. Make sure students understand that either expression can be used, and that the symbols n and x are the number of months that interest was charged on the loan.

Some students will prefer simply to multiply the number of months by 5 and add that amount to 100.

Answers

17. (a) \$150 in 10 months (b) \$200 in 20 months
    (c) \$220 in 24 months (d) \$300 in 40 months

QUESTION 18 Page 19

Encourage students to solve using one method and check using another. For example, the amount borrowed is \$500, so after the first month Jason will owe \$500 + 8% of 500 = \$540. This value becomes t_1. Each month, the constant increase is \$40; therefore, in n months, Jason will owe \$500 + 40n. So the rule in sequential notation is $t_n = 500 + 40n$.

Answers

18. (a) {\$540, \$580, \$620, \$660, \$700, \$740}
    (b) $t_n = 40n + 500$ or $y = 40x + 500$
    (c) \$1500

Think about ...

Question 18

Students can reason that Jason will pay $12 \times 40 = \$480$ interest on a loan of \$500. The annual interest rate is $\frac{480}{500} \times 100 = 96\%$. They can also reason that because the monthly rate is 8%, the annual interest rate is $12 \times 8\% = 96\%$. Have students research and report on bank loan and credit card interest rates so they will know that 96% is an extremely high rate. Some students might rightly conclude that the rate is fair because it encourages Jason to pay his sister back quickly.

Check Your Understanding

[Completion and discussion: 45 min]

QUESTION 19 — Page 19

Have students write their sequences and rules on the board so that others can check if the rules are valid.

Answer

19. Answers will vary.

QUESTION 20 — Page 19

Encourage students to use a spreadsheet for the initial arithmetic sequence and the altered sequences.

	A	B	C	D	E	F	G	H	I	J
1	Sequence	3	5	7	9	11	13	15	17	19
2	Plus 2	5	7	9	11	13	15	17	19	21
3	Minus 3	0	2	4	6	8	10	12	14	16
4	Times 10	30	50	70	90	110	130	150	170	190
5	Divided by 5	0.6	1	1.4	1.8	2.2	2.6	3	3.4	3.8
6	Squared	9	25	49	81	121	169	225	289	361
7										
8										

Encourage students to use first-level differences and graphing to identify the arithmetic sequences. With the exception of squaring, all operations will give another arithmetic sequence.

Have them report on how the operations affect the rule for the initial arithmetic sequence. For example, in the original sequence shown in the table, the relationship is $t_n = 2n + 1$. When 2 is added to each term of the original sequence, the relationship for the altered sequence is $t_n = 2n + 3$. The number added simply changes the initial term, not the common difference. Encourage students to tell what happens to the original relation when

- the same number is added to or subtracted from the terms of the original sequence
- the same number is used to multiply or divide each term of the original sequence

Answers

20. (a) to (e) Answers will vary.

Only the operation in part (e) will not give an arithmetic sequence.

Assessment

Ask students to write a notebook entry describing a way to find whether or not a sequence is an arithmetic sequence. They should also explain why the slope in a linear relation is equal to the common difference.

Challenge Yourself

This open-ended item has many possible solutions. Encourage students to use words to describe the n^{th} term, t_n, in each sequence. This will help them to find the corresponding linear relation that can be used to build each arithmetic sequence.

examples:

(a) {0, 3, 6, 9, 12} can be built by the linear relation $t_n = 3n - 3$, or $t_n = 3(n - 1)$.

(b) There are five common differences between 1 and 31. So, the common difference is $\frac{(31 - 1)}{5} = 6$. The sequence {1, 7, 13, 19, 25, 31} can be built by the linear relation $t_n = 1 + (n - 1) \times 6$ $= 6n - 5$.

(c) Start with a negative number for t_1 and simply add a constant: {−4, −2, 0, 2, 4}. This sequence is built by $t_n = 2(n - 1) - 4$ $= 2(n - 3)$.

(d) $\left\{-\frac{1}{2}, -\frac{3}{4}, -1, -\frac{5}{4}, -\frac{3}{2}\right\}$ has a common difference of $-\frac{1}{4}$. The sequence can be built by $t_n = -\frac{1}{4}(n - 1) - \frac{1}{2}$ $= -\frac{1}{4}(n + 1)$.

(e) There are 10 common differences between the first term, which is my age, (say) 17, and the eleventh term, 100. So, the common difference is $\frac{(100 - 17)}{10} = 8.3$. The sequence {17, 25.3, 33.6, ..., 100} can be built by the linear relation $t_n = 17 + (n - 1) \times 8.3$ $= 8.3n + 8.7$.

QUESTIONS 21 AND 22 Page 20

Students should have sufficient experience with arithmetic sequences to develop relations and use these to make calculations.

Answers

21. (a) The cost of printing a 400-page book is $8.00; *example*: Multiply the number of pages, n, by the printing cost of $0.02; $t_n = 0.02n$.
    (b) The cost of making a 400-page book is $12.00; *example*: Multiply the number of pages, n, by the printing cost of $0.02 and then add $4, the cost of binding; $t_n = 0.02n + 4$ or $t_n = 4 + 0.02n$.
    (c) *example*: Multiply the number of pages, n, by the printing cost of $0.03 and then add $5, the cost of binding; $t_n = 0.03n + 5$ or $t_n = 5 + 0.03n$.
22. (a) *example*: Multiply n CDs sold by the royalty of $2 per CD; $t_n = 2n$.
    (b) *example*: Multiply n CDs sold by the royalty of $2 per CD plus the cash of $10 000; $t_n = 10\ 000 + 2n$.
    (c) *example*: Multiply n CDs sold by the royalty of $2.50 per CD plus the cash of $15 000; $t_n = 15\ 000 + 2.5n$.
    (d) For (a), the earnings are 50 000 × $2 = $100 000; for (b), the earnings are $100 000 + $10 000 = $110 000; for (c), the earnings are 50 000 × $2.50 + $15 000 = $140 000.

Chapter Project

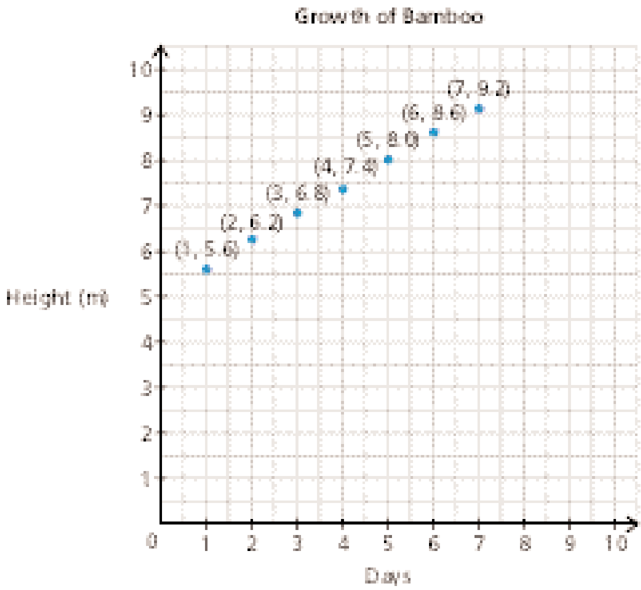
Plant and Tree Growth

Arrange to bring in examples of bamboo (such as the handle of a bamboo rake or the crosspieces of a kite) to show students that mature bamboo does have a fairly constant diameter.

Remind students to change either the measurement of 5 m to 500 cm or the measurement of 60 cm to 0.60 m.

(a) The sequence is {5.6, 6.2, 6.8, 7.4, 8, 8.6, 9.2}. Students will likely see that the sequence is an arithmetic sequence because of the constant growth of 60 cm or 0.60 m.

(b) and (c) The graph is linear—another sign that the sequence is an arithmetic sequence.



Students can use the graph to find the slope.

slope $= \frac{\text{amount of rise}}{\text{amount of run}} = \frac{6.2 - 5.6}{1} = 0.6$

Students can reason that the constant growth, slope, and common difference can be used to make a rule for bamboo growth. The initial term is 5, and $0.6n$ is repeatedly added to the initial term. The rule in symbols is $t_n = 5 + 0.6n$, where n is the number of days' growth. Students might want to use functional form and give the rule as $h = 0.6d + 5$, where h is the height of the tree in metres and d is the number of days' growth.

(d) Students can solve the equation $29 = 0.6d + 5$; $0.6d = 24$ and $d = 40$ days; after 40 days, the bamboo will be taller than 29 m. They can also repeatedly subtract 0.6 m until the original bamboo height of 5 m is obtained.

Students might also research the rapid growth rates of the vine kudzu, which has become a major problem in the United States (see **www.cptr.ua.edu/kudzu/default.htm**). They can use the growth rates to develop arithmetic sequences.

1.3 *Number Patterns: Part 2*

Suggested instruction time: 3.5 hours

Purpose of the Section

Students will solve problems with quadratic and cubic sequences by identifying and extending patterns. They will use first-, second-, and third-level sequences of differences to distinguish between arithmetic, quadratic, and cubic sequences. In some cases, they will identify a pattern and use it to create a rule for building a quadratic sequence. Geometric sequences are treated informally to show that not all sequences have an eventual common difference.

CURRICULUM OUTCOMES (SCOs)	RELATED ACTIVITIES	STUDENT BOOK
■ demonstrate an understanding of patterns that are arithmetic, power, and geometric C4	■ solve problems by identifying and extending number sequences	p. 21
	■ explore graphs of arithmetic, quadratic, and cubic sequences	p. 21
	■ use common differences to distinguish between arithmetic and power sequences	p. 22
	■ use D_n, the n^{th}-level common difference, to determine the degree of a power sequence	p. 27
	■ explore the properties of geometric sequences informally	p. 29

ASSUMED PRIOR KNOWLEDGE
■ definitions and distinguishing characteristics of linear and quadratic relations
■ using symbols to represent linear and quadratic relations

NEW TERMS AND CONCEPTS	PAGE
■ sequences of first-, second-, and third-level differences	21
■ quadratic relation or rule	22
■ quadratic sequence	22
■ cubic sequence	25
■ power sequence	27
■ geometric sequence	30

Investigation 5

Solving Problems Using Sequences of Differences

[Suggested time: 90 min] ***[Text page 21]***

Purpose

Students will explore quadratic sequences and their properties. They will use graphs, sequences of differences, and rules (relations) to distinguish between a quadratic sequence and an arithmetic sequence. They will also explore properties of figurate numbers and solve problems by extending sequences of differences.

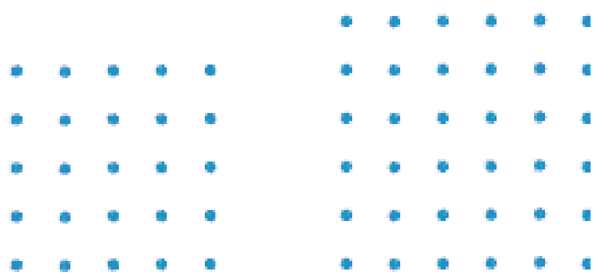
Management Suggestions

As students read the Did You Know?, mention that much of early mathematics involved the study of relationships between numbers and shapes or figures. Some students might question whether one dot could be considered a square. Explain that it is intended to represent a 1-by-1 square. Let them repeat Steps A to D using only squares with four or more dots to see the effects on the first- and second-level differences. Give students dot paper to help them show the various figurate numbers presented in the Investigation.

Procedure

Steps A and B

The next two square numbers are shown. Make sure that students see the pattern: the number of dots in each square number can be written as a square number ($1 = 1 \times 1 = 1^2$, $4 = 2 \times 2 = 2^2$, and so on).

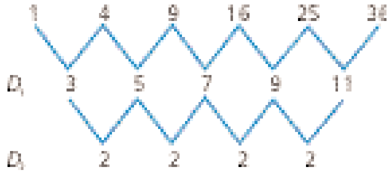


$25 = 5^2$ $36 = 6^2$

The first six terms of the sequence are {1, 4, 9, 16, 25, 36}.

Steps C and D

Have students form a tree-like diagram to show the sequences of first- and second-level differences:



Think about…

Step B

The terms in the sequence {1, 4, 9, 16, 25, 36} do not differ by a constant number as arithmetic sequences do.

Make sure students realize that each term in the sequence of second-level differences is a constant number; in this case, 2.

Investigation Questions

QUESTION 1 — Page 21

Have students look back to Investigation 3, where they found that the first-level differences in arithmetic sequences are a constant number.

Answer

1. *example*: The first-level difference was not a constant number.

QUESTION 2 — Page 21

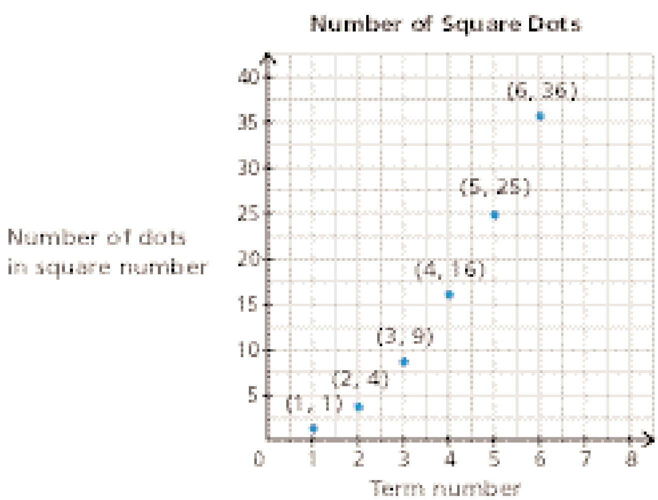
> **Note**
>
> Some students might suggest that the graph is part of a parabola, referring to their study of quadratics in *Constructing Mathematics, Book 1*.

Before students draw the graph, ask them why the graph will not be a straight line. (The graph cannot be a straight line because the number of dots from one square to the next does not change by a constant number but grows more and more.)

Students should be able to use the sequence and the shapes of the square numbers to make the rule $t_n = n^2$, where n is the term number for the n^{th} square number. Some students will simply describe the rule as *square the term number.*

Answers

2. (a) The graph has the shape of a curve whose *y*-values are increasing, but not at a uniform rate.



(b) The graph of an arithmetic sequence is a straight line. This is not.

(c) $t_n = n^2$, where n is the term number; or the n^{th} square number; or "square the term number"

(d) $50^2 = 2500$

QUESTION 3 Page 22

Remind students that they studied quadratics in *Constructing Mathematics, Book 1*. Also, refer to the note and example beside Question 2.

quadratic relation or rule— a relation or rule where the greatest value of the exponent is 2. For example, the quadratic rule $y = 2x^2 + 1$ generates the sequence {3, 9, 19, 33} for x-values of 1 to 4.

Answer

3. *example:* The rule is $t_n = n^2$ and, because the greatest exponent in the relation is 2, it represents a quadratic relation or rule.

Check Your Understanding

[Completion and discussion: 60 to 90 min]

QUESTION 4 Page 22

Assign the four rules to different groups, who can then share the results with the class. You or a student can create a spreadsheet to build the sequence and calculate D_1 and D_2.

Note that the quadratic rules have a constant second difference: in this case, 4.

Term number	1	2	3	4	5	6	7	8	9	10
$t_n = 2n^2$	2	8	18	32	50	72	98	128	162	200
D_1		6	10	14	18	22	26	30	34	38
D_2			4	4	4	4	4	4	4	4

Term number	1	2	3	4	5	6	7	8	9	10
$t_n = 2n^2 - 3$	−1	5	15	29	47	69	95	125	159	197
D_1		6	10	14	18	22	26	30	34	38
D_2			4	4	4	4	4	4	4	4

As they learned in Section 1.2, arithmetic sequences have a constant difference after one subtraction.

Term number	1	2	3	4	5	6	7	8	9	10
$t_n = 2n - 3$	−1	1	3	5	7	9	11	13	15	17
D_1		2	2	2	2	2	2	2	2	2

Note

Cubic relationships will be presented in greater depth in Investigation 6.

Suggest to the group assigned the cubic relation $t_n = n^3$ that they continue to find a third sequence of differences as shown.

Term number	1	2	3	4	5	6	7	8	9	10
$t_n = n^3$	1	8	27	64	125	216	343	512	729	1000
D_1		7	19	37	61	91	127	169	217	271
D_2			12	18	24	30	36	42	48	54
D_3				6	6	6	6	6	6	6

quadratic sequence—a sequence whose terms are numbers that can be plotted to show a quadratic relation

Management Tip

When discussing second-level differences, stress to students that D_2 must be a *non-zero* constant number to show that a sequence is quadratic.

Answers

4. (a) and (b) See the preceding tables.
   (c) The rules $t_n = 2n^2$ and $t_n = 2n^2 - 3$ are quadratic because the greatest exponent is 2.
   (d) *example*: If the second-level sequence of differences contains the same non-zero number, the original sequence is a quadratic sequence.

QUESTION 5 Page 22

Encourage students to use simple quadratic relations with whole-number coefficients to simplify the calculations. Encourage students to use graphing as well as sequences of differences to distinguish between arithmetic and quadratic sequences.

Answer

5. Answers will vary, but the quadratic sequences will have a (non-zero) constant term in the sequence of second-level differences, while the arithmetic sequences will have a (non-zero) constant term in the sequence of first-level differences.

QUESTION 6 Page 22

Students should read the definition of the quadratic sequence. They need to understand that a quadratic sequence is simply a sequence whose terms come from a quadratic rule or relation. Make a class list of the similarities and differences between arithmetic and quadratic sequences.

Answer

6. *examples:*

Arithmetic sequence

- first-level difference in an arithmetic sequence is constant
- graph of the value of the term versus the term number is part of a straight line
- the values of consecutive terms differ by a constant

Quadratic sequence

- second-level difference in a quadratic sequence is a non-zero constant
- graph of the value of the term versus the term number is part of a parabola
- the values of consecutive terms do not differ by a constant

QUESTION 7 Page 22

Students should be experienced enough to realize that the graph is likely part of a parabola and not a straight line. So, it is probably a quadratic but certainly not an arithmetic sequence. Students can confirm this by using the *y*-values in the graph to make the sequence and the sequences of differences.

Answers

7. (a) *example:* The graph is not a straight line and D_1 is not constant.
   (b) $D_1 = \{3, 5, 7, 9, 11, 13, \ldots\}$; $D_2 = \{2, 2, 2, 2, \ldots\}$; D_2 is a non-zero constant number and the sequence is a quadratic sequence.

QUESTION 8 Page 23

Students can use what they know about arithmetic and quadratic sequences to predict which sequences are quadratic and which are arithmetic. Then, they can use the rules to form the sequences. They can use either graphing or common differences to verify their predictions.

For example, parts (a) and (b) must be arithmetic sequences because an initial number increases or decreases by a constant. Sequences for parts (c) and (d) are quadratic because terms are squared or multiplied.

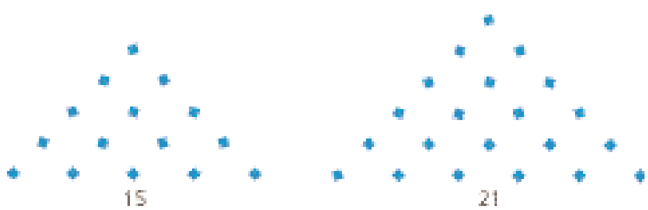
Answers

8. (a) The sequence {5, 7, 9, 11, 13, …} is an arithmetic sequence because the constant difference is 2.
   (b) The sequence {8, 5, 2, −1, −4, …} is an arithmetic sequence because the constant difference is −3.
   (c) The sequence {−2, 1, 6, 13, 22, 33, 46, 61, 78, 97, …} is a quadratic sequence because each term is squared and the second-level common difference is 2.
   (d) The sequence {2, 6, 12, 20, 30, …} is a quadratic sequence because the second-level common difference is 2.

QUESTION 9 Page 23

Before students begin to make the sequence, ask them why it is likely to be a quadratic rather than an arithmetic sequence (for example, the growth is not a constant number). The next two triangular numbers are shown.

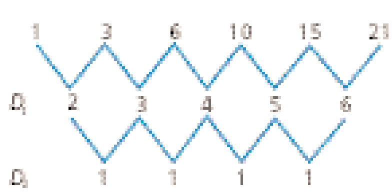
Students need to see the pattern of the number of dots from the bottom row to the top. For example, in the sixth triangular number, the number of dots can be written as 6 + 5 + 4 + 3 + 2 + 1 = 21. Have students refer to the similar pattern in the staircase problem (Question 7, page 4) and the handshake problem (Question 16, page 7) of Section 1.1.



Have students draw a tree-like diagram to show the sequences of first- and second-level differences.

The patterns that they see can be extended to make a sequence with 10 terms.

Some students might be expected only to describe how to find the 50th triangular number rather than actually calculating it. To help students develop a recursive rule for finding the 50th triangular number, suggest that they put the results in a table.



Triangular number	1	2	3	4	5	6	7	8	9	10
Number of dots	1	3	6	10	15	21	28	36	45	55

If students compare adjacent numbers in the first row and the corresponding number in the second row, they can see that $4 \times 5 = 20$ and $\frac{1}{2}$ of $20 = 10$.

Therefore, the 49th triangular number will be $\frac{49 \times 50}{2} = 1225$ and the 50th triangular number will be $\frac{50 \times 51}{2} = 1275$.

Triangular number	49	50	51
Number of dots	1225	1275	

This pattern can be developed into the rule $t_n = \frac{n(n+1)}{2}$, or $t_n = 0.5n^2 + 0.5n$. The second expression shows that the rule or relation is quadratic. Some students may need assistance in arriving at either of the above relations. For these students, describing the relation in words is acceptable.

Answers

9. (a) {1, 3, 6, 10, 15, 21, 28, 36, 45, 55}

   (b) quadratic (c) $t_n = \frac{n(n+1)}{2}$ (d) 1275 dots

QUESTION 10 Page 23

The first four terms of the pentagonal sequence are {1, 5, 12, 22}. Students can draw the fifth pentagonal number, use sequences of differences, or use patterns to find the number of dots in the fifth pentagonal number.

Have them note the geometrical pattern:

- First, four dots are added to the first pentagonal number to create the second.
- Then, seven dots are added to the second to create the third.
- Finally, 10 dots are added to the third to create the fourth.

These numbers form the sequence of first-level differences. The pattern can be extended by adding 13 dots to the fourth pentagonal number to create the fifth one, which will have a total of 35 dots. The sixth will contain 16 more dots than 35 dots for a total of 51 dots.

Answers

10. (a) {1, 5, 12, 22, 35}

    (b) quadratic because $D_2 = 3$

    (c) $t_5 = 35$ and $t_6 = 51$

QUESTION 11 Page 23

Prepare a class chart of the second-level differences for each figurate number so that students can compare.

Figurate number	D_2
triangular	1
square	2
pentagonal	3

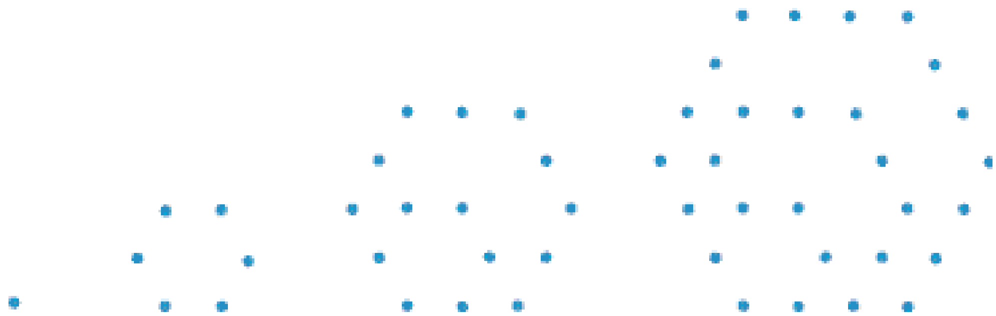
Challenge Yourself

This open-ended item will have many different solutions. Have students write their sequences on chart paper and post the charts for other students to look at. Make sure that students explain their thinking around creating the sequence.

(a) If the first term in a quadratic sequence is 3, it is likely a multiple of a square number. The easiest square number to try is 1^2. So, the first term can be 3×1^2, the second term can be 3×2^2, the third term can be 3×3^2, and so forth. The rule would be $t_n = 3n^2$ and 10 terms of the sequence would be {3, 12, 27, 48, 75, 108, 147, 192, 243, 300, …}.

(b) Students can use any quadratic relation of the form $t_n = an^2$, where $a < 0$. Therefore, a relation such as $t_n = -2n^2$ would create the quadratic sequence {−2, −8, −18, −32, −50, −72, −98, −128, −162, −200, …} with terms that are all negative.

The sequence of second-level differences for a sequence of hexagonal numbers is likely to be 4. Have students draw a series of hexagonal numbers to confirm this. The first four hexagonal numbers are shown and form part of Question 12.

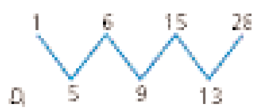


Answer

11. *example*: The differences increase by 1 for each figurate number with one more side added.

QUESTION 12 Page 24

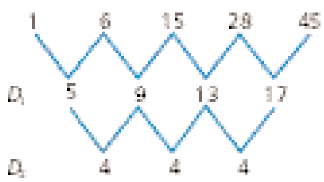
Once students see a pattern and use it to find the fifth hexagonal number, have them draw it on isometric triangular paper. Students can use sequences of differences to identify the sequence and extend it to other terms. The first four terms of the sequence are {1, 6, 15, 28}.



The next term in D_1 should be 17 because 13 + 4 = 17. Therefore, the fifth term must be 28 + 17 = 45.

Students can refer to the class chart from Question 11 to predict that D_2 must be 4:

Figurate number	D_2
triangular	1
square	2
pentagonal	3
hexagonal	**4**



Have students find the sequence of second-level differences to check this prediction.

Answers

12. (a) {1, 6, 15, 28}; 45
    (b) D_2 should be 4.
    (c) quadratic because the second difference is constant

QUESTION 13 Page 24

Have students compare their sequences for oblong numbers with the sequence from Question 8(d) and ask why they are equivalent.

Answers

13. (a) {2, 6, 12, 20}; 30, 42
    (b) quadratic because the second difference is constant